# Γ-Cohomology and the Selberg Zeta Function

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## 1 Introduction

Let $G$ be a semisimple Lie group of real rank one, $K \subset G$ be a maximal compact subgroup and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $G$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a one-dimensional subspace and $M \subset K$ be the centralizer of $\mathfrak{a}$ in $K$. Fixing a positive root system of $(\mathfrak{g}, \mathfrak{a})$ we have the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Let $(\sigma, V_\sigma) \in \hat{M}$ be an irreducible representation of $M$ and $\Gamma \subset G$ be a discrete cocompact torsion-free subgroup. Then the Selberg zeta function $Z_\Gamma(s, \sigma)$, $s \in \mathfrak{a}^*_{\mathbb{C}}$, is defined as the analytic continuation of the infinite product

$$Z_\Gamma(s,\sigma) = \prod_{[g]\in C\Gamma, [g]\neq 1, n_\Gamma(g)=1} \prod_{k=0}^{\infty} \det\left(1 - {a_g}^{-\rho-s} S^k(\mathrm{Ad}(m_g a_g)_{\mathfrak{n}}^{-1}) \otimes \sigma(m_g)\right).$$


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Here $C\Gamma$ is the set of conjugacy classes in $\Gamma$, $n_\Gamma(g)$ is maximal number $n \in \mathbb{N}$ such that $g = h^n$ for some $h \in \Gamma$ and $m_g \in M$, $a_g \in A^+$ are such that $g$ is conjugated in $G$ to $m_g a_g$. $S^k(\mathrm{Ad}(m_g a_g)_{\mathfrak{n}}^{-1})$ stands for the $k$'th symmetric power of $Ad(m_g a_g)^{-1}$ restricted to $\mathfrak{n}$ and $\rho \in \mathfrak{a}^*$ is defined by $\rho(H) := \frac{1}{2}\mathrm{tr}(\mathrm{ad}(H)_{\mathfrak{n}})$. The infinite product converges for $\mathrm{Re}(s) > \rho$. In this generality the Selberg zeta function was introduced by Fried [4]. He applied Ruelle's techniques of hyperbolic dynamics and gave a meromorphic continuation of $Z_\Gamma(s,\sigma)$.

Of main interest are the singularities of the Selberg zeta function, i.e. the poles and zeros. We distinguish the lattice of topological singularities on the negative real axis of $\mathfrak{a}^*_{\mathbb{C}}$ from the spectral singularities located (except for a finite number) on the imaginary axis $\imath\mathfrak{a}^*$ possessing a symmetry with respect to zero. In [2] we identify the topological singularities with the spectrum of a homogeneous elliptic operator on a suitable bundle over the compact dual symmetric space of $X$. The analytic continuation of this operator on the corresponding bundle over $\Gamma\backslash G/K$ is responsible for the spectral singularities of $Z_\Gamma(s,\sigma)$. Already in [15] Wakayama gave a representation theoretic description of almost all of the singularities.

Another description of the singularities in terms of $\mathfrak{n}$-cohomology was given by Juhl [7]. He employs a dynamical Lefschetz formula based on the Osborne conjecture proved by Hecht/Schmid [5]. Technical details can be found in [14]. We will recall the $\mathfrak{n}$-cohomology description in Theorem 6.7.

The parameters $(\sigma,\lambda) \in \hat{M} \times \mathfrak{a}^*_{\mathbb{C}}$ also define a principal series representation of $G$ on the Hilbert space

$$H^{\sigma,\lambda} = \{f : G \to V_\sigma \mid f(gman) = a^{\lambda-\rho}\sigma(m)^{-1}f(g),\ \forall man \in MAN, f_{|K} \in L^2\},$$

where $G$ acts by the left regular representation. By $H^{\sigma,\lambda}_{-\omega}$ we denote the space of its hyperfunction vectors.

S. Patterson [11] conjectured the relationship of the singularities of Selberg zeta functions with the cohomology of $\Gamma$ with coefficients in (subspaces of) principal series representations. In the present paper we want to prove the following theorem which settles this conjecture in the cocompact case.

**Theorem 1.1** *The cohomology $H^p(\Gamma, H^{\sigma,\lambda}_{-\omega})$ is finite dimensional for all $p \geq 0$,*

$$\chi(\Gamma, H^{\sigma,\lambda}_{-\omega}) = \sum_{p=0}^{\infty}(-1)^p \dim H^p(\Gamma, H^{\sigma,\lambda}_{-\omega}) \quad = \quad 0, \tag{1}$$

$$\chi(\Gamma, \hat{H}^{\sigma,\lambda}_{-\omega}) = \sum_{p=0}^{\infty}(-1)^p \dim H^p(\Gamma, \hat{H}^{\sigma,\lambda}_{-\omega}) \quad = \quad 0, \tag{2}$$

*and the order of $Z_\Gamma(s,\sigma)$ at $s \in \mathfrak{a}^*_{\mathbb{C}}$ can be expressed in terms of the group cohomology of $\Gamma$ with coefficients in $H^{\sigma,\lambda}_{-\omega}$ as follows :*

$$\mathrm{ord}_{s=\lambda\neq 0} Z_\Gamma(s,\sigma) \quad = \quad -\sum_{p=0}^{\infty}(-1)^p p \dim H^p(\Gamma, H^{\sigma,\lambda}_{-\omega}), \tag{3}$$

$$\mathrm{ord}_{s=0} Z_\Gamma(s,\sigma) \;=\; -\sum_{p=0}^{\infty}(-1)^p p \dim H^p(\Gamma, \hat{H}^{\sigma,0}_{-\omega}), \tag{4}$$

*where $\hat{H}^{\sigma,\lambda}_{-\omega}$ is a certain non-trivial extension of $H^{\sigma,\lambda}_{-\omega}$ with itself.*

Note that the cohomological dimension of $\Gamma$ is $n = \dim(X)$, hence $H^p(\Gamma, H^{\sigma,\lambda}_{-\omega})$ vanishes for $p > n$.

We propose a new method to study the $\Gamma$- and $\mathfrak{n}$-cohomology of the canonical globalizations of arbitrary Harish-Chandra modules $(\pi, V_{\pi,K}) \in \mathcal{HC}(\mathfrak{g}, K)$ (that is not restricted to the rank one case). Recall (see [17] Ch.11, [13], [3]) the sequence of inclusions

$$V_{\pi,K} \subset V_{\pi,\omega} \subset V_{\pi,\infty} \subset V_{\pi,-\infty} \subset V_{\pi,-\omega} \subset V_{\pi,for},$$

where $K, \omega, \infty, -\infty, -\omega, for$ stand for $K$-finite, analytic, smooth, distribution, hyperfunction and formal power series vectors of some Banach globalization of $V_{\pi,K}$. For $\omega, \infty, -\infty, -\omega$ these are smooth topological $G$-modules and the inclusions are continuous. Our main tool is a resolution (called a standard resolution, see Subsection 3.2) of $V_{\pi,-\omega}$ by $\Gamma$-acyclic and $\mathfrak{n}$-acyclic (Section 2) smooth $G$-modules given by the spaces of smooth sections of homogeneous vector bundles over $X = G/K$. The differentials of this resolution are $G$-invariant differential operators. The $\Gamma$- or $\mathfrak{n}$-cohomology of $V_{\pi,-\omega}$ is the cohomology of the subcomplex of $\Gamma$- or $\mathfrak{n}$-invariant vectors of the standard resolution. A rather simple discussion leads to the finite dimensionality and the Poincaré duality (Propositions 4.1, 4.4, 5.1, 5.2). Moreover we can show that $H^*(\mathfrak{n}, V_{\pi,\omega}) = H^*(\mathfrak{n}, V_{\pi,K})$ (Proposition 4.1). In Proposition 6.2 we construct a long exact sequence relating $H^*(\Gamma, H^{\sigma,\lambda}_{-\omega})$ with the groups $H^*(\mathfrak{n}, V_{\pi,-\omega})$ for all $\pi \in \hat{G}$ with $N_\Gamma(\pi) \neq 0$ and with the same infinitesimal character as $H^{\sigma,\lambda}$. We finish the proof of Theorem 1.1 by reduction to the description of the singularities of $Z_\Gamma(s,\sigma)$ given by Juhl [7].

In many cases we have an explicit standard resolution of the principal series representation $H^{\sigma,\lambda}_{-\omega}$. It allows us to compare $H^*(\Gamma, H^{\sigma,\lambda}_{-\omega})$ with the spectrum of the elliptic operators constructed in [2]. In the cases $G = SL(2,\mathbb{R})$ and $G = SL(2,\mathbb{C})$ this leads to a proof of the Patterson conjecture by reduction to the spectral description of the singularities of $Z_\Gamma(s,\sigma)$. It would be interesting to complete this approach in the general case.

Originally, Patterson conjectured Theorem 1.1 for the distribution globalization of the principal series. In order to study the cohomology of the distribution globalization $V_{\pi,-\infty}$ one should take the subcomplex of the standard resolution formed by the sections with at most exponential growth. Unfortunately we do not know whether this subcomplex is a resolution of $V_{\pi,-\infty}$ since we are not able to prove exactness. The essential point to show is the surjectivity of $(\Omega_G - \mu)$ ($\Omega_G$ is the Casimir operator of $G$ and $\mu \in \mathbb{C}$) on the space of sections with at most exponential growth. This is known for trivial bundles [10]. The problem in the general case is that there is still no topological Paley-Wiener theorem for the space of rapidly decreasing sections of bundles in the non-$K$-finite case. We believe that concerning the $\mathfrak{n}$- and $\Gamma$-cohomology (for cocompact $\Gamma$) there is no difference between the hyperfunction and distribution globalizations. But this difference will certainly appear when one tries to approach the more general conjecture of Patterson for finite co-volume or even more general $\Gamma$'s.

We also have a construction of a standard resolution in the higher rank case. This, examples and more applications will be the topic of another paper.

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# 2 Acyclicity Lemmas

## 2.1 $\mathfrak{n}$-acyclicity of $\mathcal{E}$

Let $E \to X$ be the homogeneous vector bundle associated to the finite dimensional representation $(\gamma, V_\gamma)$ of $K$ and let $\mathcal{E}$ be its space of smooth sections. $\mathcal{E}$ is a $\mathfrak{n}$-module with the action induced from the left regular action $g \mapsto L_g$ of $G$.

**Lemma 2.1** *We have*

$$H^p(\mathfrak{n}, \mathcal{E}) = 0, \quad \forall p \geq 1.$$

*Proof:* Using the Iwasawa decomposition $G = NAK$ we obtain

$$\mathcal{E} = [C^\infty(G) \otimes V_\gamma]^K = C^\infty(N) \otimes [C^\infty(AK) \otimes V_\gamma]^K$$

as left $\mathfrak{n}$-modules, where $\mathfrak{n}$ acts trivially on $[C^\infty(AK) \otimes V_\gamma]^K$ and the tensor products are topological ones. The $\mathfrak{n}$-cohomology complex

$$\ldots \xrightarrow{d} C^\infty(N) \otimes \Lambda^p \mathfrak{n}^* \xrightarrow{d} C^\infty(N) \otimes \Lambda^{p+1} \mathfrak{n}^* \xrightarrow{d} \ldots$$

can be identified with the de Rham complex of $N$. Since $N \cong \mathbb{R}^{\dim(\mathfrak{n})}$ via the exponential map we have a contraction $\Phi_t : N \to N$ of $N$ given by $\Phi_t(\exp(n)) := \exp(tn)$, $n \in \mathfrak{n}$, $t \in [0,1]$. The contraction $\Phi_t$ allows us to define a continuous chain contraction

$$h : C^\infty(N) \otimes \Lambda^p \mathfrak{n}^* \to C^\infty(N) \otimes \Lambda^{p-1} \mathfrak{n}^*, \quad p \geq 1$$

satisfying $dh + hd = \mathrm{id}$. The contraction $h$ extends to the tensor product with $[C^\infty(AK) \otimes V_\gamma]^K$ and the lemma follows. $\square$

## 2.2 $\mathfrak{n}$-acyclicity of $\mathcal{E}(B)$

Let $B = (\Omega_G - \lambda)^l$ for some $\lambda \in \mathbb{C}$, $l \in \mathbb{N}$, where $\Omega_G$ is the Casimir operator of $G$, and $\mathcal{E}(B) = \{f \in \mathcal{E} \mid Bf = 0\}$.

**Lemma 2.2** *We have*

$$H^p(\mathfrak{n}, \mathcal{E}(B)) = 0, \quad \forall p \geq 1.$$

*Proof:* We will use the following fact : An elliptic operator with real analytic coefficients on an analytic vector bundle over a non-compact manifold is surjective on the space of smooth sections of that vector bundle. By Lemma 2.1

$$0 \to \mathcal{E}(B) \to \mathcal{E} \xrightarrow{B} \mathcal{E} \to 0$$

is an $\mathfrak{n}$-acyclic resolution of $\mathcal{E}(B)$. Taking $\mathfrak{n}$-invariants and using the identification

$$\mathcal{E} = C^\infty(N) \otimes [C^\infty(AK) \otimes V_\gamma]^K = C^\infty(N) \otimes C^\infty(A) \otimes V_\gamma$$

we obtain the complex

$$0 \to {}^{\mathfrak{n}}\mathcal{E}(B) \to C^\infty(A) \otimes V_\gamma \xrightarrow{{}^{\mathfrak{n}}B} C^\infty(A) \otimes V_\gamma \to 0. \quad (5)$$

Here ${}^{\mathfrak{n}}B$ is the restriction of $B$ to the subspace of $\mathfrak{n}$-invariant vectors. It is a second order translation invariant differential operator on $A$. The complex (5) is again exact since ${}^{\mathfrak{n}}B$ is still elliptic. The lemma follows. □

## 2.3 $\mathfrak{n}$-acyclicity of $\mathcal{E}^{for}$ and $\mathcal{E}^{for}(B)$

Let $\mathcal{E}^{for} := \mathrm{Hom}_{\mathcal{U}(\mathfrak{k})}(\mathcal{U}(\mathfrak{g}), V_\gamma)$ be the space of formal power series sections of $\mathcal{E}$. Then $\mathcal{E}^{for}$ is a $\mathfrak{g}$-module and hence a $\mathfrak{n}$-module. Let $\mathcal{E}^{for}(B) := \{f \in \mathcal{E}^{for} \mid Bf = 0\}$.

**Lemma 2.3** *We have*

$$H^p(\mathfrak{n}, \mathcal{E}^{for}) = H^p(\mathfrak{n}, \mathcal{E}^{for}(B)) = 0, \quad \forall p \geq 1.$$

*Proof:* $\mathcal{E}^{for}$ is isomorphic to the injective $\mathfrak{n}$-module $\mathrm{Hom}(\mathcal{U}(\mathfrak{n}) \otimes \mathcal{U}(\mathfrak{a}), V_\gamma)$.

The map

$$B^* : \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} V_{\tilde{\gamma}} \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} V_{\tilde{\gamma}},$$

given by the multiplication by $B$, is injective. This can be seen by going to the graded module $Gr(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} V_{\tilde{\gamma}}) = S(\mathfrak{p}) \otimes V_{\tilde{\gamma}}$. Hence $B$ is surjective on $\mathcal{E}^{for}$. Now one can argue as in the proof of Lemma 2.2. □

## 2.4 $\Gamma$-acyclicity of $\mathcal{E}$

Let $\Gamma \subset G$ be a discrete subgroup acting properly on $X$.

**Lemma 2.4** *We have*

$$H^p(\Gamma, \mathcal{E}) = 0, \quad \forall p \geq 1.$$

*Proof:* For $p \geq 0$ let $C^p := \{f : \Gamma^{p+1} \to \mathcal{E}\}$ and $\partial : C^p \to C^{p+1}$ be defined by

$$(\partial f)(\gamma_0, \dots, \gamma_{p+1}) := \sum_{i=0}^{p+1} (-1)^i f(\gamma_0, \dots, \check{\gamma}_i, \dots, \gamma_{p+1}), \quad f \in C^p.$$

$\Gamma$ acts on $C^p$ by

$$(\gamma f)(\gamma_0, \dots, \gamma_p) = L_\gamma f(\gamma^{-1}\gamma_0, \dots, \gamma^{-1}\gamma_p), \quad \gamma \in \Gamma, f \in C^p.$$

Then $H^*(\Gamma, \mathcal{E})$ is the cohomology of the complex of $\Gamma$-invariant vectors

$$0 \to {}^\Gamma C^0 \xrightarrow{\partial} {}^\Gamma C^1 \xrightarrow{\partial} {}^\Gamma C^2 \xrightarrow{\partial} \dots .$$

Since $\Gamma$ acts properly on $X$ there is an open set $U \subset X$ such that $\{\gamma U\}_{\gamma \in \Gamma}$ is a locally finite covering of $X$. Moreover, there is a partition of unity $\{\rho_\gamma\}_{\gamma \in \Gamma}$ such that $\mathrm{supp}(\rho_\gamma) \Subset \gamma U$ and $L_{\gamma_1} \rho_\gamma = \rho_{\gamma_1 \gamma}$. Consider a cocycle $f \in {}^\Gamma C^p$, $\partial f = 0$, $p \geq 1$. Let $F \in C^{p-1}$ be defined by

$$F(\gamma_0, \dots, \gamma_{p-1})(x) = \sum_{\gamma \in \Gamma} f(\gamma_0, \dots, \gamma_{p-1}, \gamma)(x) \rho_\gamma(x).$$

The sum is finite for any $x \in X$. $F$ is $\Gamma$-invariant, $F \in {}^\Gamma C^{p-1}$. In fact

$$\begin{aligned} L_{\tilde{\gamma}} F(\tilde{\gamma}^{-1}\gamma_0, \dots, \tilde{\gamma}^{-1}\gamma_{p-1}) &= \sum_{\gamma \in \Gamma} L_{\tilde{\gamma}} f(\tilde{\gamma}^{-1}\gamma_0, \dots, \tilde{\gamma}^{-1}\gamma_{p-1}, \gamma) L_{\tilde{\gamma}} \rho_\gamma \\ &= \sum_{\gamma \in \Gamma} f(\gamma_0, \dots, \gamma_{p-1}, \tilde{\gamma}\gamma) \rho_{\tilde{\gamma}\gamma} \\ &= F(\gamma_0, \dots, \gamma_{p-1}). \end{aligned}$$

Moreover

$$\begin{aligned} (\partial F)(\gamma_0, \dots, \gamma_p) &= \sum_{i=0}^{p} (-1)^i \sum_{\gamma \in \Gamma} f(\gamma_0, \dots, \check{\gamma}_i, \dots, \gamma_p, \gamma) \rho_\gamma \\ &= \sum_{\gamma \in \Gamma} \sum_{i=0}^{p} (-1)^i f(\gamma_0, \dots, \check{\gamma}_i, \dots, \gamma_p, \gamma) \rho_\gamma \\ &= \sum_{\gamma \in \Gamma} (-1)^p f(\gamma_0, \dots, \gamma_p) \rho_\gamma \\ &= (-1)^p f(\gamma_0, \dots, \gamma_p). \end{aligned}$$

Hence $\partial(-1)^p F = f$. The lemma follows. $\Box$

## 2.5 $\mathfrak{a} \oplus \mathfrak{n}$-acyclicity of $C^{-\omega}(G)$

Let $C^{-\omega}(G)$ be the hyperfunctions on $G$ (see [12]). We consider $C^{-\omega}(G)$ as a $\mathfrak{a} \oplus \mathfrak{n}$-right module with the action induced by the right regular representation of $G$.

**Lemma 2.5** *We have*

$$H^p(\mathfrak{a}\oplus\mathfrak{n}, C^{-\omega}(G)) = 0, \quad \forall p \geq 1.$$

*Proof:* Let $\mathcal{B}$ be the sheaf of hyperfunctions on $G$. It is a sheaf of right $\mathfrak{a}\oplus\mathfrak{n}$-modules. Forming the $\mathfrak{a}\oplus\mathfrak{n}$-cohomology complex locally we obtain the complex of sheaves

$$0 \to \mathcal{K} \to \mathcal{B} \stackrel{d}{\to} \mathcal{B}\otimes\Lambda^1(\mathfrak{a}\oplus\mathfrak{n})^* \stackrel{d}{\to} \mathcal{B}\otimes\Lambda^2(\mathfrak{a}\oplus\mathfrak{n})^* \stackrel{d}{\to} \dots, \tag{6}$$

where $\mathcal{K}$ is the kernel sheaf of $d$. We claim that this complex is exact. By the left $G$-invariance it is enough to show the exactness at $e \in G$. We employ the Iwasawa decomposition $G = KAN$. Let $U \subset K$ be a small neighborhood of the identity which can be identified analytically with an open subset $V \subset \mathbb{R}^{\dim(K)}$. The $\mathfrak{a}\oplus\mathfrak{n}$-cohomology complex of $C^{-\omega}(AN)$ can be identified with the de Rham complex over $\mathbb{R}^{1+\dim(\mathfrak{n})}$. Thus

$$0 \to \mathcal{K}_{|UAN} \to \mathcal{B}_{|UAN} \stackrel{d}{\to} \mathcal{B}_{|UAN}\otimes\Lambda^1(\mathfrak{a}\oplus\mathfrak{n})^* \stackrel{d}{\to} \mathcal{B}_{|UAN}\otimes\Lambda^2(\mathfrak{a}\oplus\mathfrak{n})^* \stackrel{d}{\to} \dots.$$

is isomorphic to the sheaf version of the partial de Rham complex with hyperfunction coefficients on $V \times \mathbb{R}^{1+\dim(\mathfrak{n})}$. But this complex of sheaves is exact by Thm. 3.2 in [8]. Thus (6) is an exact complex of sheaves. The sheaf of hyperfunctions on $G$ is flabby. Hence (6) is an acyclic resolution of $\mathcal{K}$ with respect to the global section functor $\Gamma_G$. Therefore the cohomology groups of the complex of global sections are the sheaf cohomology groups of $\mathcal{K}$. On the other hand they are the $\mathfrak{a}\oplus\mathfrak{n}$-cohomology groups of $C^{-\omega}(G) = \mathcal{B}(G)$.

It remains to show that $\mathcal{K}$ is acyclic with respect to the global section functor. Let $\kappa : G = KAN \to K$ be the projection defined by the Iwasawa decomposition and $\Gamma_K$ be the global section functor for sheaves on $K$. Then there exists a spectral sequence associated to the composition $\Gamma_G = \Gamma_K \circ \kappa_*$ converging to the cohomology of $\mathcal{K}$ with second term $E_2^{p,q} = R^p\Gamma \circ R^q\kappa_*(\mathcal{K})$. The resolution (6) is also acyclic with respect to $\kappa_*$. Thus we have $R^p\kappa_*(\mathcal{K}) = 0$ for $p \geq 1$, and $R^0\kappa_*(\mathcal{K})$ can be identified with the sheaf of hyperfunctions on $K$. Since this sheaf is flabby $E_2^{p,q} \neq 0$ only for $p = q = 0$. The lemma follows. $\square$

## 2.6 $\Gamma$-acyclicity of $C^{-\omega}(G\times_M V_\sigma)$

Let $\Gamma \subset G$ be a discrete torsion-free subgroup such that $\Gamma\backslash G$ is compact. Let $(\sigma, V_\sigma)$ be a finite dimensional representation of $M$. We consider $C^{-\omega}(G\times_M V_\sigma)$ as a left $\Gamma$-module with the action induced from the left regular action of $G$.

**Lemma 2.6** *We have*

$$H^p(\Gamma, C^{-\omega}(G\times_M V_\sigma)) = 0, \quad \forall p \geq 1.$$

*Proof:* Let $\mathcal{U}$ be a finite open cover of $\Gamma\backslash G$ such that $p : G \to \Gamma\backslash G$ induces analytic diffeomorphisms of the connected components of $p^{-1}(U)$ with $U$ for all $U \in \mathcal{U}$. Let $\tilde{\mathcal{U}}$ be the open cover consisting of the connected components of the lifts of all $U \in \mathcal{U}$. Let $C^p$

be the vector space of Čech-cochains of the sheaf $\mathcal{B}_\sigma$ of hyperfunction sections of $G \times_M V_\sigma$ with respect to the cover $\tilde{\mathcal{U}}$. There is a natural $\Gamma$-action on $C^p$ given by

$$(\gamma f)(U_0, \ldots, U_p) = L_\gamma f(\gamma^{-1}U_0, \ldots, \gamma^{-1}U_p),$$

$f \in C^p$, $U_i \in \tilde{\mathcal{U}}$, $U_0 \cap \ldots \cap U_p \neq \emptyset$, $\gamma \in \Gamma$. Note that as a $\Gamma$-module $C^p = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\Gamma, \mathbb{C}) \otimes V^p$ for a certain vector space $V^p$ with the trivial $\Gamma$-action. In fact, let $S^p$ be a set of representatives with respect to $\Gamma$-translation of non-trivial intersections of $p+1$ elements of $\tilde{\mathcal{U}}$. Then we can choose $V^p := \prod_{U \in S^p} \mathcal{B}_\sigma(U)$.

Since $H^*(\Gamma, \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\Gamma, \mathbb{C})) = H^*(\{e\}, \mathbb{C})$ (Shapiro's Lemma, [1], Prop. 6.2) $C^p$ is $\Gamma$-acyclic, $H^q(\Gamma, C^p) = 0$ for all $p \geq 0$, $q \geq 1$. The Čech complex

$$0 \to C^0 \xrightarrow{\partial} C^1 \xrightarrow{\partial} C^2 \xrightarrow{\partial} \ldots \tag{7}$$

is $\Gamma$-equivariant. Since $\mathcal{B}_\sigma$ is flabby (7) is exact at all degrees $p \geq 1$. On the one hand the cohomology groups of the complex of $\Gamma$-invariant vectors in (7) are isomorphic to $H^*(\Gamma, C^{-\omega}(G \times_M V_\sigma))$. On the other hand this complex can be identified with the Čech complex of the flabby sheaf of hyperfunction sections of $\Gamma \backslash G \times_M V_\sigma$ with respect to the cover $\mathcal{U}$. The lemma follows. □

# 3 Resolutions of admissible representations

## 3.1 Constructions of differential operators

Let $(\pi, V_{\pi,K}) \in \mathcal{HC}(\mathfrak{g}, K)$. Then $V_{\pi,K}$ decomposes into a direct sum of joint generalized eigenspaces of the center $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$. Hence we may assume without loss of generality that there exist $\lambda \in \mathbb{C}$ and $l \in \mathbb{N}$ such that $B := (\Omega_G - \lambda)^l \in \mathrm{Ann}(V_{\pi,K})$.

Let $W$ be a finite dimensional $K$-stable subspace of the dual $V_{\tilde{\pi},K}$ of $V_{\pi,K}$ in the category $\mathcal{HC}(\mathfrak{g}, K)$, which generates $V_{\tilde{\pi},K}$ as a $\mathcal{U}(\mathfrak{g})$-module. Let $\tilde{W}$ be the dual $K$-module of $W$ and $E_0 \to X$ be the homogeneous vector bundle $G \times_K \tilde{W}$ and $\mathcal{E}_0$ be the space of its smooth sections. Using any globalization $V_\pi$ of $V_{\pi,K}$ (i.e. a representation of $G$ such that $V_\pi = V_{\pi,K}$) we can define an embedding

$$i : V_{\pi,K} \hookrightarrow \mathcal{E}_0 \cong [C^\infty(G) \otimes \tilde{W}]^K$$

by

$$\langle i(v)(g), w\rangle := \langle w, \pi(g^{-1})v\rangle, \qquad v \in V_{\pi,K}, w \in W, g \in G.$$

In fact, the closure of $i(V_{\pi,K})$ in $\mathcal{E}_0$ is contained in $\mathcal{E}_0(B)$ and constitutes the maximal globalization $V_{\pi,max}$ of $V_{\pi,K}$ in the sense of Schmid [13] by its very definition. Schmid's theorem identifies $V_{\pi,max}$ with the hyperfunction vectors $V_{\pi,-\omega} := ((V'_\pi)_\omega)'$ of any Banach globalization $V_\pi$ of $V_{\pi,K}$, hence $V_{\pi,-\omega}$ does not depend on the choice of the globalization $V_\pi$.

We will also consider the space $V_{\pi,for} := V^*_{\tilde{\pi},K}$ of formal power series vectors of $V_{\pi,K}$. There is an exact functor from $\mathcal{HC}(\mathfrak{g},K)$ to the category of (not necessarily $K$-finite) $(\mathfrak{g},K)$-modules which sends $V_{\pi,K}$ to $V_{\pi,for}$. Note that $V_{\pi,for} = \prod_{\gamma\in\hat{K}} V_{\pi,K}(\gamma)$.

For homogeneous vector bundles $E$ and $F$ on $X$ we denote by $D(E,F)$ the set of $G$-invariant differential operators $D : \mathcal{E} \to \mathcal{F}$.

**Proposition 3.1** *There exist homogeneous vector bundles $E_1, E_2, \ldots$ on $X$ and $G$-invariant differential operators $D_i \in D(E_i, E_{i+1})$, $i = 0, 1, \ldots$, such that the embedding $i : V_{\pi,-\omega} \hookrightarrow \mathcal{E}_0(B)$ can be extended to a (possibly infinite) exact sequence*

$$0 \to V_{\pi,-\omega} \xrightarrow{i} \mathcal{E}_0(B) \xrightarrow{D_0} \mathcal{E}_1(B) \xrightarrow{D_1} \mathcal{E}_2(B) \xrightarrow{D_2} \ldots \ . \qquad (8)$$

*This sequence remains to be exact on the level of formal power series :*

$$0 \to V_{\pi,for} \xrightarrow{i} \mathcal{E}_0^{for}(B) \xrightarrow{D_0} \mathcal{E}_1^{for}(B) \xrightarrow{D_1} \mathcal{E}_2^{for}(B) \xrightarrow{D_2} \ldots \ . \qquad (9)$$

*Proof:* Let $E$ be associated to the representation $(\gamma, V_\gamma)$ of $K$ and $\mathcal{Z}(E)$ be the image of $\mathcal{Z}(\mathfrak{g})$ in $D(E,E)$.

**Lemma 3.2** *The $\mathbb{C}[B]$-module $\mathcal{Z}(E)$ is finitely generated.*

*Proof:* By the Iwasawa decomposition we have

$$\mathcal{U}(\mathfrak{g}) = \mathfrak{n}\,\mathcal{U}(\mathfrak{g}) \oplus \mathcal{U}(\mathfrak{a}) \otimes \mathcal{U}(\mathfrak{k}) \ .$$

The projection onto the second summand gives an injective anti-homomorphism of algebras (see [16], 3.5)

$$\mathcal{U}(\mathfrak{g})^K \hookrightarrow \mathcal{U}(\mathfrak{a}) \otimes \mathcal{U}(\mathfrak{k}) \ .$$

This induces an injection

$$\mathcal{Z}(E) \overset{i}{\hookrightarrow} \mathcal{U}(\mathfrak{a}) \otimes \mathrm{End}(V_\gamma) \ .$$

The right-hand side is a finitely generated module over the principal ideal ring $\mathbb{C}[i(B)]$, hence Noetherian. We conclude that $\mathcal{Z}(E)$ is a finitely generated $\mathbb{C}[B]$-module. □

**Lemma 3.3** *We have $\mathcal{E}(B)_K \in \mathcal{HC}(\mathfrak{g},K)$.*

*Proof:* Let $(\gamma, V_\gamma)$ be the finite dimensional representation of $K$ corresponding to $E$ and $(\tilde{\gamma}, V_{\tilde{\gamma}})$ its dual. We consider the $K$-equivariant embedding

$$i : V_{\tilde{\gamma}} \hookrightarrow \widetilde{\mathcal{E}(B)_K}$$

defined by

$$i(\tilde{v})(f) := \langle \tilde{v}, f(e)\rangle \ ,$$

where we identify the fibre of $E$ at $e = [K]$ with $V_\gamma$. Let $T := \mathcal{U}(\mathfrak{g})(i(V_{\tilde{\gamma}}))$. For any $t \in T$ the dimension of $\mathcal{Z}(\mathfrak{g})t$ can be estimated by the dimension of a generating subspace of the $\mathbb{C}[B]$-module $\mathcal{Z}(E)$. Thus, by Lemma 3.2, T is a locally $Z(\mathfrak{g})$-finite and finitely generated $\mathcal{U}(\mathfrak{g})$-module. By a theorem of Harish-Chandra ([16], 3.4.7), $T \in \mathcal{HC}(\mathfrak{g}, K)$. The canonical map $\mathcal{E}(B)_K \to \tilde{T}$ is injective by the analyticity of solutions of the equation $Bf = 0$. In fact, an element in the kernel of this map would have a vanishing Taylor series at $e$. We obtain that $T \hookrightarrow \widetilde{\mathcal{E}(B)}_K$ is surjective. Thus $T = \widetilde{\mathcal{E}(B)}_K$ and $\mathcal{E}(B)_K \in \mathcal{HC}(\mathfrak{g}, K)$ since the dual of a Harish-Chandra module is a Harish-Chandra module, too ([16], 4.3.2). $\square$

**Lemma 3.4** *Let $V_{\pi,K}$ be a Harish-Chandra submodule of $\mathcal{E}(B)_K$. Then there exist a homogeneous vector bundle $F$ and an operator $D \in D(E, F)$ such that* $\ker D \cap \mathcal{E}(B)_K = V_{\pi,K}$. *We also have* $\ker D \cap \mathcal{E}(B) = V_{\pi,-\omega}$.

*Proof:* According to the proof of Lemma 3.3 there is a surjection

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} V_{\tilde{\gamma}} \to \widetilde{\mathcal{E}(B)}_K \ .$$

Let $W$ be a finite dimensional $K$-stable generating subspace of the Harish-Chandra module $V_{\pi,K}^{\perp} \subset \widetilde{\mathcal{E}(B)}_K$. Then we choose a $K$-equivariant map $\alpha$ such that the following diagram

$$\begin{array}{ccc} W & \stackrel{\alpha}{\longrightarrow} & \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} V_{\tilde{\gamma}} \\ \downarrow & & \downarrow \\ V_{\pi,K}^{\perp} & \longrightarrow & \widetilde{\mathcal{E}(B)}_K \end{array}$$

commutes. This is possible since $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} V_{\tilde{\gamma}}$ is $K$-semisimple.

We set $F := G \times_K \tilde{W}$. The map $\alpha$ can be considered as an element of

$$[\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} V_{\tilde{\gamma}} \otimes \tilde{W}]^K \cong [\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathrm{Hom}(V_\gamma, \tilde{W})]^K \ .$$

The latter space is canonically isomorphic to $D(E, F)$ via the right regular representation $R$ of $\mathcal{U}(\mathfrak{g})$ on $C^\infty(G) \otimes V_\gamma$. Thus $\alpha$ defines an element $D \in D(E, F)$. If $\alpha(w) = \sum X_i \otimes v_i$, then

$$\langle w, Df\rangle_F = \sum \langle v_i, R_{X_i} f\rangle_E \in C^\infty(G),\ w \in W, v_i \in V_{\tilde{\gamma}}, X_i \in \mathcal{U}(\mathfrak{g}) \ .$$

Let $f \in \mathcal{E}(B)$, $X \in \mathcal{U}(\mathfrak{g})$ and $w \in W$. Then we have

$$\begin{aligned} \langle w, L_X Df(1)\rangle_F &= \langle w, DL_X f\rangle_F \\ &= \sum \langle v_i, R_{X_i} L_X f(1)\rangle_E \\ &= \langle w, L_X f\rangle_{\mathcal{E}(B)} \\ &= \langle L_{X^{op}} w, f\rangle_{\mathcal{E}(B)} \ , \end{aligned} \tag{10}$$

where $X \to X^{op}$ is the anti-automorphism of $\mathcal{U}(\mathfrak{g})$ induced by the multiplication with $-1$ on $\mathfrak{g}$. By construction $Df = 0$ iff the left hand side of (10) vanishes for all $X \in \mathcal{U}(\mathfrak{g})$ and $w \in W$, while $f \in V_{\pi,-\omega}$ iff the right hand side does. The lemma follows.$\square$

In order to prove Proposition 3.1 we iterate Lemma 3.4. $D_i(\mathcal{E}_i(B)_K)$ is a Harish-Chandra submodule of $\mathcal{E}_{i+1}(B)_K$. Therefore we find a bundle $E_{i+2}$ and an operator $D_{i+1} \in D(E_{i+1}, E_{i+2})$ such that $\ker D_{i+1} \cap \mathcal{E}_{i+1}(B)_K = D_i(\mathcal{E}_i(B)_K)$. We obtain an exact sequence of Harish-Chandra modules

$$0 \to V_{\pi,K} \xrightarrow{i} \mathcal{E}_0(B)_K \xrightarrow{D_0} \mathcal{E}_1(B)_K \xrightarrow{D_1} \mathcal{E}_2(B)_K \xrightarrow{D_2} \dots \ .$$

Applying the maximal globalization functor which is exact (see [13]) we end up with (8). Analogously, we want to obtain (9) by taking formal power series vectors. This is possible according to the following lemma.

**Lemma 3.5** *For any homogeneous vector bundle $E$ we have*

$$(\mathcal{E}(B)_K)_{for} = \mathcal{E}^{for}(B) \ .$$

*Proof:* Let $E$ be associated to the $K$-representation $\underbrace{V_\gamma.}$ We have seen in the proof of Lemma 3.3 that there is a surjection $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} V_{\tilde\gamma} \to \widetilde{\mathcal{E}(B)}_K$. Hence

$$\widetilde{\mathcal{E}(B)_K} \cong \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} V_{\tilde\gamma}/(\mathcal{E}(B)_K)^\perp \ , \tag{11}$$

where $(\mathcal{E}(B)_K)^\perp$ denotes the annihilator of $\mathcal{E}(B)_K$ in $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} V_{\tilde\gamma}$. Of course, $\mathcal{E}(B)_K^\perp \supset B\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} V_{\tilde\gamma}$. We claim that $\mathcal{E}(B)_K^\perp = B\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} V_{\tilde\gamma}$. Indeed, if $\mathcal{E}(B)_K^\perp$ would be larger than $B\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} V_{\tilde\gamma}$, we could find an $f \in \mathcal{E}_K$ which on the one hand annihilates $B\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} V_{\tilde\gamma}$, that is $f \in \mathcal{E}(B)_K$, and on the other hand is non-zero on $\mathcal{E}(B)_K^\perp$. This is a contradiction. The lemma follows by dualizing (11). □

## 3.2 The standard resolution

The aim of this subsection is to extend (8) to an exact sequence of full section spaces.

**Lemma 3.6** *Let $E$, $F$ be homogeneous vector bundles on $X$ and $A \in D(E,F)$ such that $A\mathcal{E}(B) = 0$. Then $A = HB$ for some $H \in D(E,F)$.*

*Proof:* Let $E, F$ be associated to $(\gamma_i, V_{\gamma_i})$, $i = 1,2$. Since $\mathcal{E}^{for}(B)_K \cong \mathcal{E}(B)_K$ we have $A\mathcal{E}^{for}(B) = 0$. We consider the annihilator $\mathcal{E}^{for}(B)^\perp = \mathcal{E}(B)_K^\perp$ in $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} V_{\tilde\gamma_1}$. We have seen in the proof of Lemma 3.5 that this space is equal to $B\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} V_{\tilde\gamma_1}$.

We consider $A$ as an element of $[\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} V_{\tilde\gamma_1} \otimes V_{\gamma_2}]^K$. Therefore $A$ can be written as $\sum A_i \otimes v_i$, where $A_i \in \mathcal{E}^{for}(B)$ and $v_i \in V_{\gamma_2}$. Hence there exist elements $X_i \in \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} V_{\tilde\gamma_1}$ such that $A = \sum BX_i \otimes v_i$. Set

$$H := \sum X_i \otimes v_i \in [\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} V_{\tilde\gamma_1} \otimes V_{\gamma_2}]^K \cong D(E,F) \ .$$

Then $A = BH = HB$. □

Let $V_{\pi,K}, E_i, D_i$ be as in Proposition 3.1.

**Proposition 3.7** *There exist $H_i \in D(E_i, E_{i+2})$, $i \geq 0$, making the following into an exact complex:*

$$0 \to V_{\pi,-\omega} \to \mathcal{E}_0 \xrightarrow{\begin{pmatrix} D_0 \\ B \end{pmatrix}} \begin{matrix} \mathcal{E}_1 \\ \oplus \\ \mathcal{E}_0 \end{matrix} \xrightarrow{\begin{pmatrix} D_1 & H_0 \\ -B & D_0 \end{pmatrix}} \begin{matrix} \mathcal{E}_2 \\ \oplus \\ \mathcal{E}_1 \end{matrix} \xrightarrow{\begin{pmatrix} D_2 & H_1 \\ B & D_1 \end{pmatrix}} \dots \; . \tag{12}$$

We shall call (12) a standard resolution of $V_{\pi,-\omega}$.
*Proof:* In order to construct the operators $H_i$ we apply Lemma 3.6 for $A = D_{i+1}D_i$. Since $B : \mathcal{E}_i \to \mathcal{E}_i$ (as an elliptic operator with analytic coefficients on an non-compact manifold) is surjective the exactness of (12) is easily reduced to the exactness of (8). □

# 4 $\mathfrak{n}$-cohomology

## 4.1 Finite dimensionality

Let $(\pi, V_{\pi,K}) \in \mathcal{HC}(\mathfrak{g}, K)$. Recall that $H^*(\mathfrak{n}, V_{\pi,-\omega})$ carries a natural $MA$-module structure. For $\mu \in \mathfrak{a}^*_{\mathbb{C}}$ we define the generalized eigenspace

$$H^p(\mathfrak{n}, V_{\pi,-\omega})_\mu := \{\eta \in H^p(\mathfrak{n}, V_{\pi,-\omega}) \mid \exists k \text{ such that } (H - \mu(H))^k \eta = 0 \; \forall H \in \mathfrak{a}\}.$$

**Proposition 4.1** *1. The inclusion $V_{\pi,-\omega} \hookrightarrow V_{\pi,for}$ induces an isomorphism*

$$H^p(\mathfrak{n}, V_{\pi,-\omega}) \xrightarrow{\sim} H^p(\mathfrak{n}, V_{\pi,for}) \; .$$

*2.* $\dim H^p(\mathfrak{n}, V_{\pi,-\omega}) = \dim H^p(\mathfrak{n}, V_{\pi,for}) < \infty$ .

*3. Assume that $B := \Omega_G - \lambda \in \mathrm{Ann}(V_{\pi,K})$ for some $\lambda \in \mathbb{C}$. If $\mu \neq -\rho$, then $\mathfrak{a}$ acts semisimply on $H^p(\mathfrak{n}, V_{\pi,-\omega})_\mu$.*

*Proof:* According to the Lemmas 2.2, 2.3 and Proposition 3.1 $H^p(\mathfrak{n}, V_{\pi,*})$ for $* = -\omega$, $for$ is isomorphic to the cohomology of the subcomplex of $\mathfrak{n}$-invariants of (8) and (9), respectively. This together with the following lemma implies the proposition.

**Lemma 4.2** *For any homogeneous vector bundle $E \to X$ associated to $V_\gamma$ we have*

$${}^{\mathfrak{n}}\mathcal{E}(B) = {}^{\mathfrak{n}}\mathcal{E}^{for}(B) \; .$$

*Furthermore, this space is finite dimensional and consists of elements of the form*

$$f(n \exp(H) k) = \gamma(k^{-1}) \sum P_i(H) \mathrm{e}^{\lambda_i(H)}, \quad P_i \in S(\mathfrak{a}^*) \otimes V_\gamma, \lambda_i \in \mathfrak{a}^*_{\mathbb{C}} \; .$$

*If the assumption of Proposition 4.1, 3. is satisfied, then $\deg P_i \leq 1$ and $\deg P_i = 0$, whenever $\lambda_i \neq \rho$.*

*Proof:* The $\mathcal{U}(\mathfrak{a})$-module

$$^{\mathfrak{n}}\mathcal{E}^{for}(B) \cong (\widetilde{\mathcal{E}(B)_K}/\mathfrak{n}(\widetilde{\mathcal{E}(B)_K}))^*$$

is finite dimensional (see [16], Ch.4). Therefore it splits into generalized weight spaces $^{\mathfrak{n}}\mathcal{E}^{for}(B)_\mu, \mu \in \mathfrak{a}^*_{\mathbb{C}}$. $f \in {}^{\mathfrak{n}}\mathcal{E}^{for}(B)_\mu$, considered as a formal power series on $\mathfrak{a}$, satisfies the differential equations

$$(H + \mu(H))^k f = 0 \quad \forall H \in \mathfrak{a} \tag{13}$$

for a certain $k \in \mathbb{N}$. The solutions of (13) have the form

$$P(H)\mathrm{e}^{-\mu(H)}, \qquad P \in S(\mathfrak{a}^*) \otimes V_\gamma \ .$$

They extend to smooth $\mathfrak{n}$-invariant sections in $^{\mathfrak{n}}\mathcal{E}(B)$.

We are left with the proof of the last assertion. We use the following formula for the Casimir operator applied to $\mathfrak{n}$-invariant sections of $E$:

$$\Omega_G f(\exp tH) = (\mathrm{e}^{t\rho(H)}\frac{d^2}{dt^2}\mathrm{e}^{-t\rho(H)} - |\rho|^2 + \gamma(\Omega_M))f(\exp tH), \quad H \in \mathfrak{a}, |H| = 1 \ ,$$

where $\Omega_M$ is the Casimir operator of $M$ and $\gamma$ the $K$-representation defining $E$. If $f$ is of the form $f(\exp tH) = P(tH)\mathrm{e}^{-\mu(tH)}$, then $\Omega_G f = \lambda f$ implies

$$\frac{d^2}{dt^2}P(tH) - 2(\mu+\rho)(H)\frac{d}{dt}P(tH) + (\langle\mu,\mu\rangle + 2\langle\rho,\mu\rangle + \gamma(\Omega_M) - \lambda)P(tH) = 0.$$

It follows that $\deg P = 0$ if $\mu \neq -\rho$ and $\deg P \leq 1$ in the remaining case. $\Box$

## 4.2 Poincaré duality

We consider a double complex of Fréchet or dual Fréchet spaces

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & H^0 & \xrightarrow{d} & H^1 & \xrightarrow{d} & H^2 & \xrightarrow{d} & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & K^0 & \rightarrow & L^{0,0} & \xrightarrow{d} & L^{0,1} & \xrightarrow{d} & L^{0,2} & \xrightarrow{d} & \\
 & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \\
0 & \rightarrow & K^1 & \rightarrow & L^{1,0} & \xrightarrow{d} & L^{1,1} & \xrightarrow{d} & L^{1,2} & \xrightarrow{d} & \\
 & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \\
0 & \rightarrow & K^2 & \rightarrow & L^{2,0} & \xrightarrow{d} & L^{2,1} & \xrightarrow{d} & L^{2,2} & \xrightarrow{d} & \\
 & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \\
 & & \vdots & & & & & & & & 
\end{array} \ . \tag{14}$$

such that the horizontal and vertical complexes

$$
\begin{array}{l}
0 \to K^i \xrightarrow{d} L^{i,0} \xrightarrow{d} L^{i,1} \xrightarrow{d} L^{i,2} \xrightarrow{d} \dots \\
0 \to H^j \to L^{0,j} \xrightarrow{\partial} L^{1,j} \xrightarrow{\partial} L^{2,j} \xrightarrow{\partial} \dots
\end{array}
$$

are exact and $K^*, H^*$ have the induced topologies as subspaces.

**Lemma 4.3** *If the differential of the complex*

$$0 \to H^0 \xrightarrow{d} H^1 \xrightarrow{d} H^2 \xrightarrow{d} \dots \tag{15}$$

*has closed range, then so does the differential of*

$$0 \to K^0 \xrightarrow{\partial} K^1 \xrightarrow{\partial} K^2 \xrightarrow{\partial} \dots \ . \tag{16}$$

*Proof :* The double complex induces a canonical identification of the cohomologies of (15) and (16). Let $\{\eta_l\}$ be a sequence in $K^{i-1}$ such that

$$\partial\eta_l \xrightarrow{l\to\infty} \omega \in K^i \ .$$

Then $\partial\omega = 0$ and we have to show that $\omega = \partial\eta$ for some $\eta \in K^{i-1}$. Let $[\tilde{\omega}] \in H^i(H^\bullet)$ be the class corresponding to $[\omega] \in H^i(K^\bullet)$. It is enough to show that $[\tilde{\omega}] = 0$.

$\tilde{\omega}$ can be obtained by diagram chasing. We demonstrate this for $i = 2$. Let $\omega_0 \in L^{1,0}$ such that $\partial\omega_0 = \omega \in L^{2,0}$. Then

$$\partial(\omega_0 - \eta_l) \longrightarrow 0 \ .$$

There exist $f_l \in L^{0,0}$ such that

$$\omega_0 - \eta_l - \partial f_l \longrightarrow 0 \in L^{1,0} \ .$$

Applying $d$ we obtain

$$d\partial f_l = \partial d f_l \longrightarrow d\omega_0 \in L^{1,1} \ .$$

Let $\omega_1 \in L^{0,1}$ such that $\partial\omega_1 = d\omega_0$. There exist $g_l \in H^1$ with

$$\omega_1 - df_l - g_l \longrightarrow 0 \in L^{0,1}$$

and we obtain $dg_j \longrightarrow d\omega_1 =: \tilde{\omega} \in H^2$. Because of the closed range of $d$ we conclude that $[\tilde{\omega}] = 0$. $\Box$

Let $(\pi, V_{\pi,K}) \in \mathcal{HC}(\mathfrak{g}, K)$.

**Proposition 4.4** *The $\mathfrak{n}$-cohomology of $V_{\pi,-\omega}$ satisfies Poincaré duality*

$$H^p(\mathfrak{n}, V_{\pi,-\omega})^* \cong H^{\dim(\mathfrak{n})-p}(\mathfrak{n}, V_{\tilde{\pi},\omega}) \otimes \Lambda^{\dim \mathfrak{n}}\mathfrak{n}. \tag{17}$$

*Moreover,*

$$H^p(\mathfrak{n}, V_{\pi,\omega}) \cong H^p(\mathfrak{n}, V_{\pi,K}). \tag{18}$$

*Proof:* Consider a standard resolution of $V_{\pi,-\omega}$. By Lemma 2.2 the complex (8) is a $\mathfrak{n}$-acyclic resolution. Taking the $\mathfrak{n}$-cohomology complex of the complex (8) in the vertical direction we obtain a double complex of the type (14). The first vertical line becomes

$$0 \to V_{\pi,-\omega} \stackrel{d_{\mathfrak{n}}}{\to} V_{\pi,-\omega} \otimes \mathfrak{n}^* \stackrel{d_{\mathfrak{n}}}{\to} V_{\pi,-\omega} \otimes \Lambda^2\mathfrak{n}^* \stackrel{d_{\mathfrak{n}}}{\to} \dots \stackrel{d_{\mathfrak{n}}}{\to} V_{\pi,-\omega} \otimes \Lambda^{\dim(\mathfrak{n})}\mathfrak{n}^* \to 0. \qquad (19)$$

The dual of this complex is isomorphic as a complex of $MA$-modules to the $\mathfrak{n}$-cohomology complex of $V_{\tilde{\pi},\omega}$ tensored with $\Lambda^{\dim(\mathfrak{n})}\mathfrak{n}$. Here we employ the topological duality $(V_{\pi,-\omega})' = V_{\tilde{\pi},\omega}$. For (17) it is enough to show that the differential of (19) has closed range. In view of Lemma 4.3 we must show that the differential in

$$0 \to {}^{\mathfrak{n}}V_{\pi,-\omega} \to {}^{\mathfrak{n}}\mathcal{E}_0(B) \stackrel{D_0}{\to} {}^{\mathfrak{n}}\mathcal{E}_1(B) \stackrel{D_1}{\to} {}^{\mathfrak{n}}\mathcal{E}_2(B) \stackrel{D_2}{\to} \dots \qquad (20)$$

has closed range. But by Lemma 4.2 this complex is finite dimensional. The isomorphism (18) follows from the algebraic Poincaré duality

$$H^p(\mathfrak{n}, V_{\pi,K})^* \cong H^{\dim(\mathfrak{n})-p}(\mathfrak{n}, V_{\tilde{\pi},for}) \otimes \Lambda^{\dim(\mathfrak{n})}\mathfrak{n},$$

(17) and 4.1(1). $\square$

The analog of (18) for maximal nilpotent subalgebras of $\mathfrak{g}_{\mathbb{C}}$ was proved by Hecht/Taylor [6] using their extension to the analytic category of the localization techniques of Beilinson/Bernstein.

# 5 Γ-cohomology

## 5.1 Finite dimensionality

Let $(\pi, V_{\pi,K}) \in \mathcal{HC}(\mathfrak{g}, K)$ and $\Gamma \subset G$ be a discrete torsion free cocompact subgroup.

**Proposition 5.1** *We have*

$$\dim H^p(\Gamma, V_{\pi,-\omega}) < \infty, \quad \forall p \geq 0.$$

*Proof:* Let

$$0 \to V_{\pi,-\omega} \to \mathcal{E}_0 \xrightarrow{\begin{pmatrix} D_0 \\ B \end{pmatrix}} \begin{array}{c}\mathcal{E}_1\\ \oplus \\ \mathcal{E}_0\end{array} \xrightarrow{\begin{pmatrix} D_1 & H_0 \\ -B & D_0 \end{pmatrix}} \begin{array}{c}\mathcal{E}_2\\ \oplus \\ \mathcal{E}_1\end{array} \xrightarrow{\begin{pmatrix} D_2 & H_1 \\ B & D_1 \end{pmatrix}} \dots$$

be a standard resolution of $V_{\pi,-\omega}$. By Lemma 2.4 the cohomology of the subcomplex of $\Gamma$-invariant vectors is isomorphic to $H^*(\Gamma, V_{\pi,-\omega})$.

For any homogeneous vector bundle $E \to X$ the space of smooth $\Gamma$-invariant sections ${}^{\Gamma}\mathcal{E}$ can be identified with the space of smooth sections of $\Gamma\backslash E \to \Gamma\backslash X$. Since $B$ is elliptic and normal with respect to the canonical $L^2$-structure on ${}^{\Gamma}\mathcal{E}$ and $\Gamma\backslash X$ is compact we can split ${}^{\Gamma}\mathcal{E} = {}^{\Gamma}\mathcal{E}(B) \oplus {}^{\Gamma}\mathcal{E}(B)^{\perp}$.

We do this splitting for all ${}^\Gamma\mathcal{E}_i$ entering the standard resolution. We obtain a complex which is a direct sum of an exact complex built from the ${}^\Gamma\mathcal{E}_i(B)^\perp$ and a complex of finite dimensional vector spaces

$$0 \to {}^\Gamma\mathcal{E}_0(B) \xrightarrow{\begin{pmatrix} D_0 \\ B \end{pmatrix}} \begin{array}{c} {}^\Gamma\mathcal{E}_1(B) \\ \oplus \\ {}^\Gamma\mathcal{E}_0(B) \end{array} \xrightarrow{\begin{pmatrix} D_1 & H_0 \\ -B & D_0 \end{pmatrix}} \begin{array}{c} {}^\Gamma\mathcal{E}_2(B) \\ \oplus \\ {}^\Gamma\mathcal{E}_1(B) \end{array} \xrightarrow{\begin{pmatrix} D_2 & H_1 \\ B & D_1 \end{pmatrix}} \dots \tag{21}$$

The cohomology of the latter complex is isomorphic to $H^*(\Gamma, V_{\pi,-\omega})$. The proposition follows. □

## 5.2 Poincaré duality

**Proposition 5.2** *The* $\Gamma$*-cohomology of* $V_{\pi,\pm\omega}$ *satisfies Poincaré duality*

$$H^p(\Gamma, V_{\pi,-\omega})^* \cong H^{n-p}(\Gamma, V_{\tilde\pi,\omega}),$$

*where* $n = \dim(X)$*.*

*Proof:* Note that $V_{\pi,-\omega}$ is a Fréchet representation of $\Gamma$ and $V_{\pi,\omega}$ is its topological dual. Since $\Gamma\backslash X$ is an oriented compact manifold we can find a finite oriented simplicial complex $P$ being homeomorphic to $\Gamma\backslash X$. From a barycentric subdivision of $P$ we can construct two oriented simplicial complexes $\mathbf{K}, \tilde{\mathbf{K}}$ being homeomorphic to $P$ such that $\tilde{\mathbf{K}}$ is dual to $\mathbf{K}$ (see [9], Ch.VI). I.e., for any oriented $p$-simplex $\sigma^p \subset \mathbf{K}$ there is an unique oriented $n-p$-simplex $\tilde\sigma^{n-p} \subset \tilde{\mathbf{K}}$ such that $\sigma^p \cap \tilde\sigma^{n-p}$ is a barycenter of a simplex of $P$ and the algebraic intersection number is 1. Note that $\Gamma\backslash X$ has the homotopy type of the classifying space $B\Gamma$. The representation $V_{\pi,-\omega}$ gives rise to a local system over $\mathbf{K}$. We form the associated cochain complex

$$0 \to C^0 \xrightarrow{\partial} C^1 \xrightarrow{\partial} C^2 \xrightarrow{\partial} \dots, \tag{22}$$

where

$$C^p = \bigoplus_{\sigma^p \in \mathbf{K}} V_{\pi,-\omega}.$$

We have identified the space of constant sections of the local system over $\sigma^p$ with the fibre over $\sigma^p \cap \tilde\sigma^{n-p}$, where $\tilde\sigma^{n-p}$ is dual to $\sigma^p$. We shall see that the topological dual of the complex (22) is exactly the cochain complex

$$0 \to \tilde C^0 \xrightarrow{\tilde\partial} \tilde C^1 \xrightarrow{\tilde\partial} \tilde C^2 \xrightarrow{\tilde\partial} \dots$$

associated to $\tilde{\mathbf{K}}$ and the local system induced by $V_{\pi,\omega}$. We define the pairing $C^p \otimes \tilde C^{n-p} \to \mathbb{C}$ as follows : The summand $V_{\pi,-\omega} \subset C^p$ corresponding to $\sigma^p$ is paired nontrivially with the summand $V_{\pi,\omega} \subset \tilde C^{n-p}$ corresponding to the dual simplex $\tilde\sigma^{n-p}$. In order to show that

Poincaré duality holds it is enough to show that the differential $\partial : C^p \to C^{p+1}$ has a closed range for all $p \geq 0$.

Let

$$0 \to L^0 \stackrel{d}{\to} L^1 \stackrel{d}{\to} L^2 \stackrel{d}{\to} \dots$$

be a standard resolution of $V_{\pi,-\omega}$. Each $L^p$ is a Fréchet representation of $\Gamma$ and gives rise to a local system over $\mathbf{K}$ and to a complex

$$0 \to C^{0,p} \stackrel{\partial}{\to} C^{1,p} \stackrel{\partial}{\to} C^{2,p} \stackrel{\partial}{\to} \dots.$$

These complexes fit together to a double complex of Fréchet spaces, where $C^{j,p} \to C^{j,p+1}$ is induced by $(-1)^p d$. On the one hand the cohomology with respect to $d$ is concentrated in the zero degree since the standard resolution is exact. It yields exactly the complex (22). On the other hand the cohomology with respect to $\partial$ is also concentrated in the zero degree since the $L^p$, $p \geq 0$, are $\Gamma$-acyclic by Lemma 2.4. It gives the complex

$$0 \to {}^\Gamma\mathcal{E}_0 \xrightarrow{\begin{pmatrix} D_0 \\ B \end{pmatrix}} \begin{array}{c} {}^\Gamma\mathcal{E}_1 \\ \oplus \\ {}^\Gamma\mathcal{E}_0 \end{array} \xrightarrow{\begin{pmatrix} D_1 & H_0 \\ -B & D_0 \end{pmatrix}} \begin{array}{c} {}^\Gamma\mathcal{E}_2 \\ \oplus \\ {}^\Gamma\mathcal{E}_1 \end{array} \xrightarrow{\begin{pmatrix} D_2 & H_1 \\ B & D_1 \end{pmatrix}} \dots .$$

The differential of the latter complex has closed range. Applying Lemma 4.3 we conclude that the differential of (22) has closed range, too. $\square$

In a similar manner one can also prove a Poincaré duality theorem using the hermitian dual.

# 6 Γ-cohomology of the principal series

## 6.1 A long exact sequence

Let $(\sigma, V_\sigma) \in \hat{M}$ and $\lambda \in \mathfrak{a}^*_\mathbb{C}$. We define the representation $\sigma_\lambda$ of $MAN$ on $V_{\sigma_\lambda} := V_\sigma$ by $\sigma_\lambda(man) = a^{\rho-\lambda}\sigma(m)$. Consider $C^{-\omega}(G) \otimes V_{\sigma_\lambda}$ as a right $\mathfrak{a} \oplus \mathfrak{n}$-module. Here $\mathfrak{a}$ acts on both $C^{-\omega}(G)$ and $V_{\sigma_\lambda}$. The $\mathfrak{a} \oplus \mathfrak{n}$-cohomology complex of $C^{-\omega}(G) \otimes V_{\sigma_\lambda}$

$$0 \to C^{-\omega}(G)\otimes V_{\sigma_\lambda} \to C^{-\omega}(G)\otimes V_{\sigma_\lambda}\otimes\Lambda^1(\mathfrak{a}\oplus\mathfrak{n})^* \to C^{-\omega}(G)\otimes V_{\sigma_\lambda}\otimes\Lambda^2(\mathfrak{a}\oplus\mathfrak{n})^* \to \dots \quad (23)$$

is exact in all degrees $p \geq 1$ by Lemma 2.5. In fact, as a right $\mathfrak{a}\oplus\mathfrak{n}$-module $C^{-\omega}(G) \otimes V_{\sigma_\lambda}$ can be identified with $C^{-\omega}(G) \otimes V_\sigma$ with the trivial $\mathfrak{a}\oplus\mathfrak{n}$-action on $V_\sigma$. The complex (23) admits an $M$-action induced from the right regular action of $M$ on $C^{-\omega}(G)$ and $\sigma$. Since $M$ is compact, the subcomplex

$$0 \to C^{-\omega}(G\times_M V_{\sigma_\lambda}) \to C^{-\omega}(G\times_M(V_{\sigma_\lambda}\otimes\Lambda^1(\mathfrak{a}\oplus\mathfrak{n})^*)) \to C^{-\omega}(G\times_M(V_{\sigma_\lambda}\otimes\Lambda^2(\mathfrak{a}\oplus\mathfrak{n})^*)) \to \dots \quad (24)$$

of $M$-invariants is still acyclic in all degrees $p \geq 1$. The complex (24) admits a left $G$-action induced from the left regular action on $C^{-\omega}(G)$. By Lemma 2.6 (24) is a $\Gamma$-acyclic resolution of its zero'th cohomology for any cocompact torsion free discrete subgroup

$\Gamma \subset G$. But the zero'th cohomology of (24) is the space of $MAN$-invariant hyperfunctions in $C^{-\omega}(G)\otimes V_{\sigma_\lambda}$ and can be identified as a $G$-module with the maximal globalization $H^{\sigma,\lambda}_{-\omega}$ of the principal series. We have therefore proved the following result.

**Proposition 6.1** *The cohomology of*

$$0 \to C^{-\omega}(\Gamma\backslash G \times_M V_{\sigma_\lambda}) \to C^{-\omega}(\Gamma\backslash G \times_M (V_{\sigma_\lambda} \otimes \Lambda^1(\mathfrak{a}\oplus\mathfrak{n})^*)) \to \\ \to C^{-\omega}(\Gamma\backslash G \times_M (V_{\sigma_\lambda} \otimes \Lambda^2(\mathfrak{a}\oplus\mathfrak{n})^*)) \to \dots \tag{25}$$

*is isomorphic to* $H^*(\Gamma, H^{\sigma,\lambda}_{-\omega})$.

Let $\chi_{\sigma,\lambda}$ be the infinitesimal character of $H^{\sigma,\lambda}_{-\omega}$. Let $T$ be the finite set of equivalence classes of irreducible unitary representations of $G$ with infinitesimal character $\chi_{\sigma,\lambda}$. The right regular representation of $G$ on $L^2(\Gamma\backslash G)$ decomposes as

$$L^2(\Gamma\backslash G) = \bigoplus_{\pi\in\hat{G}}^{\text{Hilbert}} N_\Gamma(\pi)V_\pi, \tag{26}$$

where $N_\Gamma(\pi) \in \mathbb{N}$ and $N_\Gamma(\pi)V_\pi := \bigoplus_{i=1}^{N_\Gamma(\pi)} V_\pi$. Note that $H^*(\mathfrak{n}, V_{\pi,-\omega})$ carries a natural $MA$-module structure. Let $H$ be a unit vector in $\mathfrak{a}$.

**Proposition 6.2** *There exists a long exact sequence*

$$\dots \xrightarrow{h} \bigoplus_{\pi\in T} N_\Gamma(\pi)[H^p(\mathfrak{n}, V_{\pi,-\omega})\otimes V_{\sigma_\lambda}]^M \to H^{p+1}(\Gamma, H^{\sigma,\lambda}_{-\omega}) \to \\ \to \bigoplus_{\pi\in T} N_\Gamma(\pi)[H^{p+1}(\mathfrak{n}, V_{\pi,-\omega})\otimes V_{\sigma_\lambda}]^M \xrightarrow{h} \dots, \tag{27}$$

*where* $h : H^p(\mathfrak{n}, V_{\pi,-\omega})\otimes V_{\sigma_\lambda} \to H^p(\mathfrak{n}, V_{\pi,-\omega})\otimes V_{\sigma_\lambda}$ *is the action of* $H$ *induced by the* $\mathfrak{a}$*-module structures of the* $\mathfrak{n}$*-cohomology and of* $V_{\sigma_\lambda}$.

*Proof:* From (26) we obtain a decomposition

$$C^{-\omega}(\Gamma\backslash G \times_M (V_{\sigma_\lambda} \otimes \Lambda^p(\mathfrak{a}\oplus\mathfrak{n})^*)) = \bigoplus_{\pi\in T} N_\Gamma(\pi)[V_{\pi,-\omega}\otimes V_{\sigma_\lambda}\otimes \Lambda^p(\mathfrak{a}\oplus\mathfrak{n})^*]^M \oplus R^p,$$

where $R^p = [C^{-\omega}(\Gamma\backslash G)_\perp \otimes V_{\sigma_\lambda} \otimes \Lambda^p(\mathfrak{a}\oplus\mathfrak{n})^*]^M$ and $C^{-\omega}(\Gamma\backslash G)_\perp$ is the space of hyperfunction vectors of the subrepresentation $(\oplus_{\pi\in T} N_\Gamma(\pi)V_\pi)^\perp \subset L^2(\Gamma\backslash G)$ of the right regular representation. The complex (25) decomposes into a direct sum of two subcomplexes

$$0 \to \bigoplus_{\pi\in T} N_\Gamma(\pi)[V_{\pi,-\omega}\otimes V_{\sigma_\lambda}]^M \to \bigoplus_{\pi\in T} N_\Gamma(\pi)[V_{\pi,-\omega}\otimes V_{\sigma_\lambda}\otimes\Lambda^1(\mathfrak{a}\oplus\mathfrak{n})^*]^M \to \\ \to \bigoplus_{\pi\in T} N_\Gamma(\pi)[V_{\pi,-\omega}\otimes V_{\sigma_\lambda}\otimes\Lambda^2(\mathfrak{a}\oplus\mathfrak{n})^*]^M \to \dots \tag{28}$$

and

$$\mathcal{R}^\bullet : 0 \to R^0 \xrightarrow{d} R^1 \xrightarrow{d} R^2 \xrightarrow{d} \dots . \tag{29}$$

**Lemma 6.3** *The complex (29) is exact.*

*Proof:* The set of infinitesimal characters $\chi_\pi$ of representations $\pi \in \hat{G}$ with $N_\Gamma(\pi) \neq 0$ has no accumulation points. Thus we find a finite set $A_i \in \mathcal{Z}(\mathfrak{g})$ and $\epsilon > 0$ such that for any $\pi \in \hat{G}$ with $N_\Gamma(\pi) \neq 0$ and $\chi_\pi \neq \chi_{\sigma,\lambda}$ there is some $i(\pi)$ with $|\chi_\pi(A_{i(\pi)}) - \chi_{\sigma,\lambda}(A_{i(\pi)})| \geq \epsilon$. The center $\mathcal{Z}(\mathfrak{g})$ acts on the complexes (25), (29). The induced action on $H^*(\Gamma, H^{\sigma,\lambda}_{-\omega})$ has the infinitesimal character $\chi_{\sigma,\lambda}$. Let $[\alpha] \in H^p(\mathcal{R}^\bullet)$, $\alpha \in R^p$, $d\alpha = 0$ . Then for $A \in \mathcal{Z}(\mathfrak{g})$ we have $A[\alpha] = [A\alpha] = \chi_{\sigma,\lambda}(A)[\alpha]$. Thus $(A - \chi_{\sigma,\lambda}(A))\alpha = d\beta(A)$ for some $\beta(A) \in R^{p-1}$. Note that $R^{p-1}$ has a further decomposition as a topological direct sum

$$R^{p-1} = \bigoplus_{\pi\in\hat{G},\chi_\pi\neq\chi_{\sigma,\lambda}}^{\text{topological}} \bigoplus_{j=1}^{N_\Gamma(\pi)} [V_{\pi,-\omega} \otimes V_{\sigma_\lambda} \otimes \Lambda^{p-1}(\mathfrak{a} \oplus \mathfrak{n})^*]^M .$$

We decompose

$$\beta(A) = \bigoplus_{\pi\in\hat{G},\chi_\pi\neq\chi_{\sigma,\lambda}} \bigoplus_{j=1}^{N_\Gamma(\pi)} \beta^j_\pi(A).$$

Define

$$\gamma := \bigoplus_{\pi\in\hat{G},\chi_\pi\neq\chi_{\sigma,\lambda}} \bigoplus_{j=1}^{N_\Gamma(\pi)} \frac{\beta^j_\pi(A_{i(\pi)})}{\chi_\pi(A_{i(\pi)}) - \chi_{\sigma,\lambda}(A_{i(\pi)})}.$$

Then $\gamma \in R^{p-1}$ and $\alpha = d\gamma$. The lemma follows. $\Box$

From Lemma 6.3 follows that $H^*(\Gamma, H^{\sigma,\lambda}_{-\omega})$ is isomorphic to the cohomology of (28). We have $\Lambda^p(\mathfrak{a} \oplus \mathfrak{n})^* = \Lambda^p\mathfrak{n}^* \oplus \Lambda^{p-1}\mathfrak{n}^* \otimes \mathfrak{a}^*$. We identify $\mathfrak{a}^* \cong \mathbb{C}$ using $H$. Then (28) is isomorphic to a finite direct sum of subcomplexes

$$0 \to [V_{\pi,-\omega} \otimes V_{\sigma_\lambda}]^M \xrightarrow{\begin{pmatrix} d \\ H \end{pmatrix}} \begin{matrix} [V_{\pi,-\omega} \otimes V_{\sigma_\lambda} \otimes \mathfrak{n}^*]^M \\ \oplus \\ [V_{\pi,-\omega} \otimes V_{\sigma_\lambda}]^M \end{matrix} \xrightarrow{\begin{pmatrix} d & 0 \\ -H & d \end{pmatrix}}$$
$$\longrightarrow \begin{matrix} [V_{\pi,-\omega} \otimes V_{\sigma_\lambda} \otimes \Lambda^2\mathfrak{n}^*]^M \\ \oplus \\ [V_{\pi,-\omega} \otimes V_{\sigma_\lambda} \otimes \mathfrak{n}^*]^M \end{matrix} \xrightarrow{\begin{pmatrix} d & 0 \\ H & d \end{pmatrix}} \dots . \quad (30)$$

Here any $\pi \in T$ contributes to (28) with $N_\Gamma(\pi)$ copies of (30). The differential $d$ is the differential of the $\mathfrak{n}$-cohomology complex. A standard argument of homological algebra now gives the long exact sequence (27). This finishes the proof of the proposition. $\Box$

A similar argument applies in the higher rank case. We obtain a spectral sequence relating the $\mathfrak{n}$-cohomology of maximal globalizations of admissible representations and the $N_\Gamma(\pi)$ with the Γ-cohomology of the principal series representation associated to a minimal parabolic subgroup.

$H^*(\mathfrak{n}, V_{\pi,-\omega})$ is a direct sum of generalized $H$-eigenspaces. By Proposition 4.1 for $\lambda \neq 0$ the generalized $\lambda(H) - \rho(H)$-eigenspace is in fact $\ker(H - \lambda(H) + \rho(H))$.

**Corollary 6.4** *For $\lambda \neq 0$, $p \geq 0$ we have*

$$H^p(\Gamma, H^{\sigma,\lambda}_{-\omega}) \cong \bigoplus_{\pi\in T} N_\Gamma(\pi) \left[(H^p(\mathfrak{n}, V_{\pi,-\omega}) \oplus H^{p-1}(\mathfrak{n}, V_{\pi,-\omega})) \otimes V_{\sigma_\lambda}\right]^{M,H},$$

*where $[.]^{M,H}$ stands for $M$-invariant vectors in the kernel of $H$.*

By Lemma 4.2 $H^2$ acts semisimply on $H^*(\mathfrak{n}, V_{\pi,-\omega})$ for all $\pi$. We modify the differential of the complex (23) replacing the action of $H$ by $H^2$. Then it remains still exact in all degrees $p \geq 1$. Taking $M$-invariants we obtain a corresponding modification of (24). Its zeroth cohomology is, as a $G$-module, a non-trivial extension $\hat{H}^{\sigma,\lambda}_{-\omega}$ of $H^{\sigma,\lambda}_{-\omega}$ with itself :

$$0 \to H^{\sigma,\lambda}_{-\omega} \to \hat{H}^{\sigma,\lambda}_{-\omega} \to H^{\sigma,\lambda}_{-\omega} \to 0.$$

Arguing as above we obtain

**Proposition 6.5** *There exists a long exact sequence*

$$\ldots \xrightarrow{h^2} \bigoplus_{\pi\in T} N_\Gamma(\pi)[H^p(\mathfrak{n}, V_{\pi,-\omega}) \otimes V_{\sigma_\lambda}]^M \to \tag{31}$$

$$\to H^{p+1}(\Gamma, \hat{H}^{\sigma,\lambda}_{-\omega}) \to \bigoplus_{\pi\in T} N_\Gamma(\pi)[H^{p+1}(\mathfrak{n}, V_{\pi,-\omega}) \otimes V_{\sigma_\lambda}]^M \xrightarrow{h^2} \ldots,$$

*where $h^2 : H^p(\mathfrak{n}, V_{\pi,-\omega}) \otimes V_{\sigma_\lambda} \to H^p(\mathfrak{n}, V_{\pi,-\omega}) \otimes V_{\sigma_\lambda}$ is the action of $H^2$ induced by the $\mathfrak{a}$-module structures of the $\mathfrak{n}$-cohomology and of $V_{\sigma_\lambda}$.*

**Corollary 6.6** *For $p \geq 0$ we have*

$$H^p(\Gamma, \hat{H}^{\sigma,\lambda}_{-\omega}) \cong \bigoplus_{\pi\in T} N_\Gamma(\pi) \left[(H^p(\mathfrak{n}, V_{\pi,-\omega}) \oplus H^{p-1}(\mathfrak{n}, V_{\pi,-\omega})) \otimes V_{\sigma_\lambda}\right]^{M,H^2},$$

*where $[.]^{M,H^2}$ stands for $M$-invariant vectors in the kernel of $H^2$.*

## 6.2 Proof of the Patterson Conjecture

The assertions (1) and (2) follow immediately from 6.4 and 6.6.

We now recall the description of the singularities of $Z_\Gamma(s,\gamma)$ given in [7]. Let $w \in N_K(\mathfrak{a})$ represent the non-trivial element of the Weyl group $W \cong \mathbb{Z}_2$ of $(\mathfrak{g}, \mathfrak{a})$. Then $w(\lambda) = -\lambda$, $\lambda \in \mathfrak{a}^*_\mathbb{C}$. Moreover if $(\sigma, V_\sigma) \in \hat{M}$ we let $\sigma^w$ be the representation of $M$ on $V_\sigma$ given by $\sigma^w(m) = \sigma(wmw^{-1})$. Thus $(\sigma_\lambda)^w = (\sigma^w)_{-\lambda}$. For $(\pi, V_{\pi,K}) \in \mathcal{HC}(\mathfrak{g}, K)$ let

$$\tilde{\chi}(\pi,\sigma,\lambda) := \sum_{p=0}^{\dim(\mathfrak{n})} (-1)^p \dim[H^p(\mathfrak{n}, V_{\pi,K}) \otimes V_{(\sigma_\lambda)^w}]^{M,H^\infty},$$

where $[.]^{M,H^\infty}$ stands for the $M$-invariants in the generalized 0-eigenspace of $H$. By Lemma 4.2 it is enough to take the kernel of $H^2$.

**Theorem 6.7 (Juhl, [7] Thm. 7.2.1)**

$$ord_{s=\lambda}Z_\Gamma(s,\sigma) = (-1)^{\dim(\mathfrak{n})} \sum_{\pi\in\hat{G}} N_\Gamma(\pi)\tilde{\chi}(\pi,\sigma,\lambda). \tag{32}$$

By Proposition 4.4 we have for all $(\pi, V_{\pi,K}) \in \mathcal{HC}(\mathfrak{g}, K)$

$$H^p(\mathfrak{n}, V_{\pi,K}) = H^{\dim(\mathfrak{n})-p}(\mathfrak{n}, V_{\pi,-\omega})^* \otimes \Lambda^{\dim(\mathfrak{n})}\mathfrak{n}^*.$$

Let

$$\chi(\pi,\sigma,\lambda) := \sum_{p=0}^{\dim(\mathfrak{n})} (-1)^p \dim[H^p(\mathfrak{n}, V_{\pi,-\omega}) \otimes V_{(\tilde{\sigma}^w)_\lambda}]^{M,H^2}.$$

Then $\chi(\pi,\sigma,\lambda) = (-1)^{\dim(\mathfrak{n})}\tilde{\chi}(\pi,\sigma,\lambda)$. It is easy to see from Lemma 6.3 that if $\chi_\pi \neq \chi_{\tilde{\sigma}^w,\lambda}$, then $N_\Gamma(\pi)\chi(\pi,\sigma,\lambda) = 0$. Thus we can restrict the summation in (32) over the finite set $\tilde{T} := \{\pi \in \hat{G} \mid \chi_\pi = \chi_{\tilde{\sigma}^w,\lambda}\}$. Set

$$\chi_1(\Gamma, V_{\pi,-\omega}) := \sum_{p=0}^{\infty}(-1)^p p \dim(H^p(\Gamma, V_{\pi,-\omega})).$$

From Corollary 6.4 and 6.6 it follows that

$$\begin{aligned} \sum_{\pi\in\tilde{T}} N_\Gamma(\pi)\chi(\pi,\sigma,\lambda) &= -\chi_1(\Gamma, H^{\tilde{\sigma}^w,\lambda}_{-\omega}), \qquad \lambda \neq 0 \\ \sum_{\pi\in\tilde{T}} N_\Gamma(\pi)\chi(\pi,\sigma,0) &= -\chi_1(\Gamma, \hat{H}^{\tilde{\sigma}^w,0}_{-\omega}). \end{aligned} \tag{33}$$

We apply now identities in the Γ-cohomology of principal series representations for different parameters $(\sigma, \lambda)$ in order to reduce (3) and (4) to (33) and (32). It is known that the singularities of $Z_\Gamma(s,\sigma)$ are on $\mathfrak{a}^* \cup \imath\mathfrak{a}^*$.

We first discuss the case $\sigma = \sigma^w$. If $\lambda \in \mathfrak{a}^*$, then by applying Poincaré duality twice (one times with the complex linear dual and then with the hermitian dual) we get

$$\chi_1(\Gamma, H^{\tilde{\sigma}^w,\lambda}_{-\omega}) = (-1)^{\dim(\mathfrak{n})+1}\chi_1(\Gamma, H^{\sigma,-\lambda}_{\omega}) = \chi_1(\Gamma, H^{\sigma,\lambda}_{-\omega}).$$

By (32) we see that (3) holds for $\lambda \in \mathfrak{a}^* \setminus \{0\}$. In a similar fashion we obtain (4). If $\lambda \in \imath\mathfrak{a}^* \setminus \{0\}$ we have again by Poincaré duality $\chi_1(\Gamma, H^{\tilde{\sigma}^w,\lambda}_{-\omega}) = \chi_1(\Gamma, H^{\sigma,-\lambda}_{-\omega})$. By (32) and the unitary equivalence $H^{\sigma,\lambda} = H^{\sigma,-\lambda}$ equation (3) also holds for imaginary $\lambda$. Now we consider the case $\sigma \neq \sigma^w$. Then all singularities of $Z_\Gamma(s,\sigma)$ are on $\imath\mathfrak{a}^*$. For $\lambda \in \imath\mathfrak{a}^* \setminus \{0\}$ we have the unitary equivalence $H^{\sigma,\lambda} = H^{\sigma^w,-\lambda}$. Using this and again Poincaré duality we obtain $\chi_1(\Gamma, H^{\tilde{\sigma}^w,\lambda}_{-\omega}) = \chi_1(\Gamma, H^{\sigma,\lambda}_{-\omega})$. Now (3) follows for $\lambda \neq 0$ from (32). If $\lambda = 0$ we apply Poincaré duality twice to obtain $\chi_1(\Gamma, \hat{H}^{\tilde{\sigma}^w,0}_{-\omega}) = \chi_1(\Gamma, \hat{H}^{\sigma^w,0}_{-\omega})$. Then (4) follows since $\mathrm{ord}_{s=0}Z_\Gamma(s,\sigma) = \mathrm{ord}_{s=0}Z_\Gamma(s,\sigma^w)$ (see [2]). This finishes the proof of Theorem 1.1.□

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