A fractal SUSY-QM model and the Riemann hypothesis

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Abstract

The Riemann's hypothesis (RH) states that the nontrivial zeros of the Riemann zeta-function are of the form $s = 1/2 + i\lambda_n$. Hilbert-Polya argued that if a Hermitian operator exists whose eigenvalues are the imaginary parts of the zeta zeros, λ_n's, then the RH is true. In this paper a fractal supersymmetric quantum mechanical (SUSY-QM) model is proposed to prove the RH. It is based on a quantum inverse scattering method related to a fractal potential given by a Weierstrass function (continuous but nowhere differentiable) that is present in the fractal analog of the CBC (Comtet, Bandrauk, Campbell) formula in SUSY QM. It requires using suitable fractal derivatives and integrals of irrational order whose parameter β is one-half the fractal dimension of the Weierstrass function. An ordinary SUSY-QM oscillator is constructed whose eigenvalues are of the form $\lambda_n = n\pi$, and which coincide with the imaginary parts of the zeros of the function $\sin(iz)$. This sine function obeys a trivial analog of the RH. A review of our earlier proof of the RH based on a SUSY QM model whose potential is related to the Gauss-Jacobi theta series is also included. The spectrum is given by $s(1-s)$ which is real in the critical line (location of the nontrivial zeros) and in the real axis (location of the trivial zeros).

1 Introduction

Riemann's outstanding hypothesis (RH) that the non-trivial complex zeros of the zeta-function $\zeta(s)$ must be of the form $s' = 1/2 \pm i\lambda_n$, is one of most important open problems in pure mathematics. The zeta-function has a relation with the number of prime numbers less than a given quantity and the zeros of zeta are deeply connected with the distribution of primes [1]. References [2, 3, 4] are devoted to the mathematical properties of the zeta-function.

The RH has also been studied from the point of view of physics (e.g., [5, 6, 7, 8, 9]). For example, the spectral properties of the λ_n's are associated with the random statistical fluctuations of the energy levels (quantum chaos) of a classical chaotic system [8]. Montgomery [10] has shown that the two-level correlation function of the distribution of the λ_n's coincides with the expression obtained by Dyson with the help of random matrices corresponding to a Gaussian unitary ensemble. Planat [11] has found a link between RH and the called $1/f$ noise. Wu and Sprung [12] have numerically shown that the lower lying non-trivial zeros can be related to the eigenvalues of a Hamiltonian having a fractal structure. For a recent and nice discussion on several quantum hamiltonians related to the prime numbers distribution and the zeros of the zeta function see the work by Rosu [13]. Since the literature on the topic is rather extensive we refer the reader to a nice review of zeta-related papers which can be found in Ref. [14].

Scattering theory on real and p-adic symmetric spaces produces S-matrices involving the Riemann zeta function [15]. Scattering on the noncompact finite area fundamental domain of $SL(2,Z)$ on the real hyperbolic plane was studied long ago by Fadeev and Pavlov [16], and more recently by Planat and Perrine [17] within the context of the deep arithmetical properties underlying the physics of $1/f$ noise.

Scattering matrix s-wave amplitudes for scattering in the Poincare disk can be expressed in the form [18]:

$$S = \frac{c(k)}{c(-k)} = \frac{\zeta(ik)\zeta(1-ik)}{\zeta(1+ik)\zeta(-ik)} = e^{i2\delta_0(k)}, \tag{1}$$

where $c(k)$ are the Harish-Chandra c-functions (Jost functions). The Jost functions are defined whether the space is symmetric or not, and whether a suitable potential is introduced or not. s-wave scattering by a potential with a cutoff have been recently studied by [19] where the complex zeros of the Jost functions yield the complex poles of the S-matrix that are located on a horizontal line (below the real axis) and which can be mapped into the critical line of zeros of the Riemann zeta function. They represent resonances. For example, in the case of s-wave scattering in the hyperbolic plane (Poincare disk) one can show that the complex-poles of the S-matrix correspond to the nontrivial zeros when,

$$k_n = i(1/2 + i\lambda_n). \tag{2}$$

Hence, a Wick rotation of the Riemann critical line yields the complex momenta associated with the double poles of the S-matrix above; *i.e.* the double zeros of the denominator. If one could find a physical reason why the complex double poles of the S-matrix should always occur in complex-conjugate pairs:

$$-ik_n = (1 + ik_n)^* = 1 - ik_n^* \Rightarrow k_n = i(1/2 + i\lambda_n), \tag{3}$$

one would have found a physical proof of the RH. Pigli has discussed why scattering theory on real and p-adic systems involving the Riemann zeta function belong to a wide class of integrable models that can be unified into an Adelic integrable systems whose S-matrix involves the Dirichlet, Langlands, Shimura, L-functions.

In this work we will also invoke an integrability property associated with the quantum inverse scattering problem associated with a (fractal) SUSY QM model that yields the one-to-one correspondence among the imaginary parts of the zeta zeros λ_n with the phases α_n of a fractal Weierstrass function. One could also consider a stochastic process having an underlying hidden Parisi-Sourlas supersymmetry, as the effective motion of a particle in a potential which can be expanded in terms of an infinite collection of p-adic harmonic oscillators (See in [20]). But in this case we will focus entirely on a fractal SUSY QM model with a judicious fractal potential.

Wu and Sprung have made a very insightful and key remark pertaining the conundrum of constructing a one-dimensional integrable and time-reversal quantum Hamiltonian to model the imaginary parts of the zeros of zeta as an eigenvalue problem. This riddle of merging chaos with integrability is solved by choosing a fractal local potential that captures the chaotic dynamics inherent with the zeta zeros.

By a Fractal SUSY QM model studied here, we do not mean systems with fractional supersymmetries which are common in the string and M-theory literature, but a Hamiltonian operator that admits a factorization into two factors involving fractional derivative operators whose irrational order is one-half of the fractal dimension of the fractal potential. A model of fractal spin has been constructed by Wellington da Cruz [21] in connection to the fractional quantum Hall effect based on the filling factors associated with the Farey fractions. The self-similarity properties of the Farey fractions are widely known to posses remarkable fractal properties [22]. For further details of the validity of the RH based on the Farey fractions and the Franel-Landau shifts we refer to the literature on the zeta function.

In previous work [20, 23, 24] we have already explored some possible strategies which could lead to a solution of the problem. The last one was based on the relation of the non-trivial zeros of the ζ-function with the orthogonality of eigenfunctions of the appropriately chosen operator (see also [25, 26, 27]). We have not assumed any *ad-hoc* symmetries like conformal invariance, but in fact, we shown why the $t \to 1/t$ symmetry is in direct correlation with the $s' \to 1-s'$ symmetry of the Riemann's fundamental identity $Z(s') = Z(1-s')$, the function Z is the Riemann fundamental function defined in (12). This was the clue of our proposal to proof the RH.

In this work we illustrate the method in [24] by applying it to the study of the zeros of a very simple function, the $\sin(is)$. The proof that the zeros of $\sin(is)$ are given by $0 + iy_n = 0 + in\pi$ is trivial. Nevertheless, one can still furnish another proof following the same steps as the proof of the RH in [24].

The contents of this work are the following. In section 2.1 we review the proof of the RH [24] and concentrate in section 2.2 on a SUSY QM model whose potential is related to the Gauss-Jacobi theta series. The inner product of the eigenfunctions $\psi_s(t), \psi_s(1/t)$ of the partner (non-Hermitian) Hamiltonians H_A, H_B is given by $Z(as+b)$ while their spectrum is $s(1-s)$ which happens to be real only in the critical line (location of the nontrivial zeta zeros) and in the real axis (location of the trivial zeta zeros). In section 2.3 some important remarks about the Eisenstein series and our approach are made. In section 3 we present a proof of the SRH, the sine version of the RH. In section 4 we consider the ordinary SUSY QM model solution of the SRH and finally we construct the fractional (fractal) supersymmetric quantum mechanical (SUSY-QM) model whose spectrum yields the imaginary parts λ_n of the nontrivial zeros of zeta. It is based on a quantum inverse scattering method related to a fractal potential given by a Weierstrass function (continuous but nowhere differentiable) that is present in the fractal analog of the CBC (Comtet, Bandrauk, Campbell) formula in SUSY QM. It requires using suitable fractal derivatives and integrals of irrational order whose parameter β is one-half the fractal dimension of the Weierstrass function.

2 Nontrivial ζ's zeros as an orthogonality relation

Our proposal is based on finding the appropriate operator D_1

$$D_1 = -\frac{d}{d\ln t} + \frac{dV}{d\ln t} + k, \tag{4}$$

such that its eigenvalues s are complex-valued, and its eigenfunctions are given by

$$\psi_s(t) = t^{-s+k} e^{V(t)}. \tag{5}$$

D_1 is not self-adjoint since its eigenvalues are complex valued numbers s. We also define the operator dual to D_1 as follows,

$$D_2 = \frac{d}{d\ln t} + \frac{dV}{d\ln t} + k, \tag{6}$$

that is related to D_1 by the substitution $t \to 1/t$ and by noticing that

$$\frac{dV(1/t)}{d\ln(1/t)} = -\frac{dV(1/t)}{d\ln t},$$

where $V(1/t)$ is not equal to $V(t)$.

Since $V(t)$ can be chosen arbitrarily, we choose it to be related to the Bernoulli string spectral counting function, given by the Jacobi theta series,

$$e^{2V(t)} = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t^l} = 2\omega(t^l) + 1. \tag{7}$$

This choice is justified in part by the fact that Jacobi's theta series ω has a deep connection to the integral representations of the Riemann zeta-function [28].

Latter arguments will rely also on the following related function defined by Gauss,

$$G(1/x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2/x} = 2\omega(1/x) + 1, \tag{8}$$

where $\omega(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$. Then, our V is such that $e^{2V(t)} = G(t^l)$. We defined x as t^l. We call $G(x)$ the Gauss-Jacobi theta series (GJ).

Thus we have to consider a family of D_1 operators, each characterized by two real numbers k and l which can be chosen arbitrarily. The measure of integration $d\ln t$ is scale invariant. Let us mention that D_1 is also invariant under scale transformations of t and $F = e^V$ since $dV/(d\ln t) = d\ln F/(d\ln t)$. In [25] only one operator D_1 is introduced with the number $k = 0$ and a different (from ours) definition of F.

We define the inner product as follows,

$$\langle f|g\rangle = \int_0^\infty f^* g \frac{dt}{t}. \tag{9}$$

Based on this definition the inner product of two eigenfunctions of D_1 is

$$\begin{aligned} \langle \psi_{s_1}|\psi_{s_2}\rangle &= \alpha \int_0^\infty e^{2V} t^{-s_{12}+2k-1} dt \\ &= \frac{2\alpha}{l} Z\left[\frac{2}{l}(2k - s_{12})\right], \end{aligned} \tag{10}$$

where we have denoted

$$s_{12} = s_1^* + s_2 = x_1 + x_2 + i(y_2 - y_1),$$

used the expressions (7) and (31) and noticed that

$$\langle s_1|s_2\rangle = \langle 1/2 + i0|s_{12} - 1/2\rangle.$$

Thus, the inner product of ψ_{s_1} and ψ_{s_2} is equivalent to the inner product of ψ_{s_o} and ψ_s, where $s_o = 1/2 + i0$ and $s = s_{12} - 1/2$. Constant α is to be appropriately chosen so that the inner product in the critical domain is semi-positively definite. The integral is evaluated by introducing a change of variables $t^l = x$ (which gives $dt/t = (1/l)dx/x$) and using the result provided by the equation (8), given in Karatsuba and Voronin's book [2]. Function Z in (31) can be expressed in terms of the Jacobi theta series, $\omega(x)$ defined by (7) (see [3]),

$$\begin{aligned} \int_0^\infty \sum_{n=1}^\infty e^{-\pi n^2 x} x^{s/2-1} dx &= \\ &= \int_0^\infty x^{s/2-1}\omega(x)dx \\ &= \frac{1}{s(s-1)} + \int_1^\infty [x^{s/2-1} + x^{(1-s)/2-1}]\omega(x)dx \\ &= Z(s) = Z(1-s). \end{aligned} \tag{11}$$

where

$$Z(s) \equiv \pi^{-s/2}\Gamma(s/2)\zeta(s), \tag{12}$$

and obeys the functional relation $Z(s) = Z(1-s)$.

Since the right-hand side of (11) is defined for all s this expression gives the analytic continuation of the function $Z(s)$ to the entire complex s-plane [3]. In this sense the fourth "=" in (11) is not a genuine equality. Such an analytic continuation transforms this expression into the inner product, defined by (10).

A recently published report by Elizalde, Moretti and Zerbini [27] (containing comments about the first version of our paper [29]) considers in detail the consequences of the analytic continuation implied by equation (11). One of the consequences is that equation (10) loses the meaning of being a scalar product. Arguments by Elizalde *et al.* [27] show that the construction of a genuine inner product is impossible.

Therefore from now on we will loosely speak of a "scalar product" realizing that we do not have a scalar product as such. The crucial problem is whether there are zeros outside the critical line (but still inside the critical strip) and not the interpretation of equation (10) as a genuine inner product. Despite this, we still rather loosely refer to this mapping as a scalar product. The states still have a real norm squared, which however need not to be positive-definite.

Here we must emphasize that our arguments do not rely on the validity of the zeta-function regularization procedure [30], which precludes a rigorous interpretation of the right hand side of (11) as a scalar product. Instead, we can simply replace the expression "scalar product of ψ_{s_1} and ψ_{s_2}" by the map S of complex numbers defined as

$$\begin{array}{rrcl} S: & \mathcal{C}\otimes\mathcal{C} & \rightarrow & \mathcal{C} \\ & (s_1, s_2) & \mapsto & S(s_1, s_2) = -Z(as+b), \end{array} \tag{13}$$

where $s = s_1^* + s_2 - 1/2$ and $a = -2/l; b = (4k-1)/l$. In other words, our arguments do not rely on an evaluation of the integral $\langle\psi_{s_1}|\psi_{s_2}\rangle$, but only on the mapping $S(s_1, s_2)$, defined as the finite part of the integral (10). The kernel of the map $S(s_1, s_2) = -Z(as+b)$ is given by the values of s such that $Z(as+b) = 0$, where $\langle s_1|s_2\rangle = \langle s_o|s\rangle$ and $s_o = 1/2 + i0$. Notice that $2b + a = 4(2k-1)/l$. We only need to study the "orthogonality" (and symmetry) conditions with respect to the "vacuum" state s_o to prove the RH from our theorem 2. By symmetries of the "orthogonal" states to the "vacuum" we mean always the symmetries of the kernel of the S map.

The "inner" products are trivially divergent due to the contribution of the $n = 0$ term of the GJ theta series in the integral (10). From now on, we denote for "inner" product in (10) and (13) as the finite part of the integrals by simply removing the trivial infinity. We shall see in the next section, that this "additive" regularization is in fact compatible with the symmetries of the problem.

2.1 Three theorems and a proof of the RH

In our approach, the RH emerges as a consequence of the symmetries of the orthogonal states to the "vacuum" state ψ_{s_o}. To this end we prove now the first theorem:

Th. 1. If a and b are such that $2b+a=1$, the symmetries of all the states ψ_s orthogonal to the "vacuum" state are preserved by any map S, equation (13), which leads to $Z(as+b)$

Proof: If the state associated with the complex number $s=x+iy$ is orthogonal to the "vacuum" state and the "scalar product" is given by $Z(as+b)=Z(s')$, then the Riemann zeta-function has zeros at $s'=x'+iy'$, s'^*, $1-s'$ and $1-s'^*$.

If we equate $as+b=s'$, then $as^*+b=s'^*$. Now, $1-s'$ will be equal to $a(1-s)+b$, and $1-s'^*$ will be equal to $a(1-s^*)+b$, if, and only if, $2b+a=1$. Therefore, all the states ψ_s orthogonal to the "vacuum" state, parameterized by the complex number $1/2+i0$, will then have the same symmetry properties with respect to the critical line as the nontrivial zeros of zeta.

Notice that our choice of $a=-2/l$ and $b=(4k-1)/l$ is compatible with this symmetry if k and l are related by $l=4(2k-1)$. Conversely, if we assume that the orthogonal states to the "vacuum" state have the same symmetries of $Z(s)$, then a and b must be related by $2b+a=1$. This results in a very specific relation between k and l, obtained from $a+2b=1$ for a,b real. It is clear that a map with arbitrary values of a and b does not preserve the above symmetries.

Th. 2 . The $s'\to 1-s'$ symmetry of the Riemann nontrivial zeros and the $t\to 1/t$ symmetry of the "inner" products, are concatenated with the $s\to\beta-s$ symmetry of the "orthogonal" states to a "vacuum" state $s_o=\beta/2+i0$, for any real β.

Proof: Gauss has shown that [31],

$$G(1/x)=x^{1/2}\,G(x), \tag{14}$$

where the Jacobi series $G(x)$ is defined by equation (7). (14) implies that one can always find a β, such that $\psi_s(1/t)=\psi_{\beta-s}(t)$ for all values of s if, and only if, $2k-\beta=l/4$. Due to (k,l) are real, this forces β be a real. In terms of (a,b) this relation becomes, $1=a(2\beta-1)+b$, that when $\beta=1$ gives the known relation $1=a+2b$.

Then, invariance of the "inner" product under the inversion symmetry, $t\to 1/t$ follows by adopting a standard regularization procedure of removing the infinities, which yields the well defined finite parts: $\langle\psi_{1/2+i0}(t)|\psi_s(t)\rangle = \langle\psi_{1/2+i0}(1/t)|\psi_s(1/t)\rangle = \langle\psi_{1/2+i0}(t)|\psi_{1-s}(t)\rangle = -Z(s') = -Z(s'')$. If this invariance under inversion holds for all values of s and due to the fact that $s'\neq s''$ (except for the trivial case when $1-s=s$, $s=1/2$) the only consistent solution, for all values of s, has to be $s''=1-s'$ due to Riemann's fundamental identity $Z(s')=Z(1-s')$.

The origins of the symmetry $t\to 1/t$ in the scalar product $\langle s_o|s\rangle$ stem from the invariance of the integral (10,11) (modulo the infinities) under the $x\to 1/x$ transformation. Such invariance is translated as an invariance under $s'\to 1-s'$, based on the Gauss-Jacobi relation. We have not assumed any *ad hoc* symmetries, like conformal invariance, without justifying their origins. We are basing everything in the fundamental relation $Z(s')=Z(1-s')$, therefore our symmetry $t\to 1/t$ is well justified.

Th. 3 . From the symmetries of theorem 2, one can easily show that $a+2b = 1$. Now we will demonstrate how by choosing a continuous family of operators with $l = 8k-4$ (*i.e.* $a+2b=1$), the RH is a direct consequence of the fact that the states orthogonal to the "vacuum" state have the same symmetry properties as the zeros of ζ-function.

The RH is a direct consequence of the assumption that the kernel of the map $Z(as+b)$ has the same symmetry properties as the zeros of zeta. This means that the values of s such that $Z(as+b) = 0$; i.e. the states "orthogonal" to the "vacuum" state $s_o = 1/2 + i0$, are symmetrically distributed with respect to the critical line and come in multiplets of four arguments s, $1-s$, s^*, $1-s^*$.

Proof: Due to the analytic properties of the function $Z(as+b) = Z(s')$ it follows from theorem 1 that such symmetry conditions are satisfied if and only if: $a(k,l) + 2b(k,l) = 1$, implying that $l = 8k-4$ from which in turn follows that: $s' = a(k,l)s + b(k,l) = a(k,l)(s-1/2) + 1/2$, so their real parts satisfy: $x' = 1/2 + a(k,l)(x-1/2)$.

Let us assume that the putative zeros are located on the vertical lines parallel to the Riemann critical line, which can be written as $s'_m = x'_m + iy'_{mn}$ where m labels the particular vertical line, and n labels the height of such zero along the vertical line. Hence, for a fixed value of x_m, the value of the real part x'_m can be continuously changed by continuously changing (k,l), since $a = -2/l$. And vice versa, x'_m can be held fixed whereas the location of x_m can be continuously changed as one varies a. If we assume that the *vertical* lines of orthogonal states and zeros belong to a discrete set of lines, instead of a continuum of lines, this requires that $x_m = 1/2$ is the only consistent value that the orthogonal states can have for their real parts. From this follows that $x'_m = 1/2$ is the only consistent and possible value which the real part of the zeros of zeta can have. Therefore, RH follows directly from the latter conclusion.

However, since the location of the y' values of the zeros varies along the critical Riemann line, these arguments, of course, cannot provide for the location of the imaginary parts of the zeros. If one has $y' = ay$, it is clear that the fixed points (for all values of a) will be $y = y' = 0$, which is clearly incompatible with the fact that there are no zeros of the function $Z(s')$ located in the real horizontal axis and that there are an infinity of nontrivial zeros of zeta (in the critical line) whose imaginary parts are distinct from zero!.

Concluding, if, and only if, one assumes a discrete set of vertical lines of zeros, for all values of a, this can be satisfied provided the orthogonal states have for their real parts the value $x = 1/2$, which yields $x' = 1/2$ as the only possible solution which is the RH and the orthogonality conditions among the eigenfunctions $\psi_s(t)$ have a one-to-one correspondence with the zeta zeros. However, this argument does not, cannot, yield the correct varying values of y. A complete argument which determines both the x and the y values follows next.

2.2 The zeros from supersymmetric quantum mechanics

A more satisfactory argument to prove the RH can be found following the Hilbert-Polya proposal. We will see also that this symmetry of the "vacuum", in the particular case $\beta = 1$, is also compatible with the isospectral property of the two partner Hamiltonians,

$$H_A = D_2 D_1 = \left[\frac{d}{d\ln t} - \frac{dV(1/t)}{d\ln(1/t)} + k\right]\left[-\frac{d}{d\ln t} + \frac{dV(t)}{d\ln t} + k\right], \tag{15}$$

and

$$H_B = D_1 D_2 = \left[-\frac{d}{d\ln t} + \frac{dV(t)}{d\ln t} + k\right]\left[\frac{d}{d\ln t} - \frac{dV(1/t)}{d\ln(1/t)} + k\right]. \tag{16}$$

Notice that $V(1/t) \neq V(t)$ and for this reason D_2 is not the "adjoint" of D_1. Operators defined on the half line do not admit an adjoint extension, in general. Hence, the partner Hamiltonians H_A, H_B are not (self-adjoint) Hermitian operators like it occurs in the construction of SUSY QM. Consequently their eigenvalues are not real in general.

Nevertheless one can show by inspection that if, and only if, $\psi_s(1/t) = \psi_{1-s}(t)$ then both partner Hamiltonians are isospectral (like in SUSY QM) whose spectrum is given by $s(1-s)$ and the corresponding eigenfunctions are,

$$H_A\psi_s(t) = s(1-s)\psi_s(t). \quad H_B\psi_s(1/t) = s(1-s)\psi_s(1/t). \tag{17}$$

Firstly by a direct evaluation one can verify,

$$D_1\psi_s(t) = s\psi_s(t) \text{ and } D_2\psi_s(1/t) = s\psi_s(1/t), \tag{18}$$

i.e. $\psi_s(t)$ and $\psi_s(1/t)$ are eigenfunctions of the D_1 and D_2 operators respectively with complex eigenvalue s. Secondly, if, and only if, the condition $\psi_s(1/t) = \psi_{1-s}(t)$ is satisfied, then it follows that:

$$\begin{array}{ccccccc} H_B\psi_s(1/t) & = & D_1 D_2\psi_s(1/t) & = & sD_1\psi_s(1/t) & = \\ sD_1\psi_{1-s}(t) & = & s(1-s)\psi_{1-s}(t) & = & s(1-s)\psi_s(1/t), & \end{array} \tag{19}$$

meaning that $\psi_s(1/t)$ is an eigenfunction of H_B with $s(1-s)$ eigenvalue.

$$\begin{array}{ccccccc} H_A\psi_s(t) & = & D_2 D_1\psi_s(t) & = & sD_2\psi_s(t) & = \\ sD_2\psi_{1-s}(1/t) & = & s(1-s)\psi_{1-s}(1/t) & = & s(1-s)\psi_s(t), & \end{array} \tag{20}$$

meaning that $\psi_s(t)$ is an eigenfunction of H_A with $s(1-s)$ eigenvalue.

Therefore, under condition $\psi_s(1/t) = \psi_{1-s}(t)$ the non-Hermitian partner Hamiltonians are isospectral. The spectrum is $s(1-s)$. The operators H_A and H_B are quadratic in derivatives like the Laplace-Beltrami operator and involve two generalized dilatation operators D_1 and D_2. Notice the most important results of this section:

1. On the critical Riemann line, because $\mathcal{R}e(s) = 1/2 \rightarrow 1 - s = s*$, the eigenvalues are real since $s(1-s) = ss^*$ is real. The function $Z(s)$ is also real on the critical line as a result of $Z(s) = Z(1-s) = Z(s^*)$.

2. On the real line, the eigenvalues $s(1-s)$ are trivially real.

Therefore, the spectrum $s(1-s)$ of the two partner (non-Hermitian) Hamiltonians is real-valued when s falls in the critical line (location of nontrivial zeros) and when s falls in the real line (location of trivial zeros). Hence, the SUSY QM model yields the precise location of the lines of the trivial and nontrivial zeros of zeta!. Notice the similarity of these results with the eigenvalues of the Laplace Beltrami operator in the hyperbolic plane associated with the chaotic billiard living on a surface of constant negative curvature. In that case the Selberg zeta function (which obeys the RH) played a crucial role [6].

The states $\psi_s(t)$ constitute an over-complete basis. An orthonormal discrete and complete basis can be found, when $s_n = 1/2 + i\lambda_n$, by simply recurring to the orthogonality conditions of the states $|s_n\rangle$ with respect to the "ground" or "vacuum" state $|s_o\rangle = |1/2 + i0\rangle$. By starting with $|0(t)\rangle = |s_o\rangle$ the first orthonormal state is $|1(t)\rangle = a_{11}|s_1\rangle$. The normalization condition $\langle s_1|s_1\rangle = a_{11}Z[1/2 + i0] = 1$ will yield the real value of the coefficient a_{11}. The function $Z(1/2 + iy)$ is real for all values of y. Iterating this procedure gives:

$$|n(t)\rangle = \sum_{m=1}^{n} a_{nm}|\psi_{s_m}(t)\rangle, \tag{21}$$

for all $s_m = 1/2 + i\lambda_m$ such that $m = 1, 2, ...n$. The real coefficients a_{mn} are determined by imposing the orthogonality and normalization conditions:

$$\langle m'(t)|m(t)\rangle = \delta_{m'm}. \tag{22}$$

In this fashion the discrete and complete orthonormal basis $|1(t)\rangle$, $|2(t)\rangle$, ... $|n(t)\rangle$, $|n+1(t)\rangle$, all the way to $n = \infty$ of states is constructed in terms of the eigenfunctions $\psi_s(t), \psi_s(1/t)$ of the two partner H_A, H_B Hamiltonians associated with a SUSY QM model and which is entirely based on the locations of the nontrivial zeros of zeta in the critical line.

To sum up, the inversion properties under $t \rightarrow 1/t$ of the eigenfunctions of the infinite family of differential operators, $D_1^{(k,l)}(t)$ and $D_2^{(k,l)}(1/t)$, compatible with the existence of an invariant "vacuum", are responsible for the isospectral condition of the partner non-Hermitian Hamiltonians, H_A and H_B, like it occurs in SUSY QM. The spectrum $s(1-s)$ is real in the critical line (location of the nontrivial zeros) and in the real line (location of the trivial zeros). The quantum inverse scattering problem associated with a fractal SUSY QM model which yields the imaginary parts of the nontrivial zeros consistent with the Hilbert-Polya proposal to prove the RH will be studied in the next sections. The supersymmetric ground state is precisely that associated with $s_o = 1/2 + i0$. Rosu has recast our SUSY QM wave equations into a transparent SUSY QM form [13].

2.3 A remark on Eisenstein series

Let's emphazise the importance of the Eisenstein series $E(s,z)$ being the two-dimensional analog of what we did in section 2.1.

Using the fundamental function $Z(s) = Z(1-s)$ one constructs the function $I(s,z)$ defined as $I(s,z) = Z(2s)E(s,z)$ which obeys the same functional relation as the $Z(s)$ (See [32]). Notice the crucial $2s$ argument inside the Z. It reads:

$$I(s,z) = I(1-s,z). \tag{23}$$

Note that it is the function $I(s,z)$ and not the $E(s,z)$ that obeys the same functional relation as $Z(s)$.

The function $I(s,z)$ admits also a theta series representation, and the eigenfunctions of the 2-D Laplacian in the hyperbolic plane are given by the $E(s,z)$. The eigenvalue problem for the two-dimensional Laplacian in the hyperbolic plane is:

$$y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)E(s,z) = -s(1-s)E(s,z), \tag{24}$$

where $z = x+iy$ (notice the eigenvalues). One has used the Laplace-Beltrami differential operator in non-Euclidean geometries. The hyperbolic metric is conformally flat and for this reason the hyperbolic Laplacian must be conformal to the ordinary Laplacian in flat spaces. This explains the prefactor of y^2 in front of the ordinary Laplacian.

Since the Laplacian is two-dimensional, this means that the Eisenstein series $E(s,z)$ are the 2-D version (s, z are both complex and independent) of our eigenfunctions $\psi(s,t)$ of the 1-D Laplacian-like operator obeying:

$$H_A\psi(s,t) = s(1-s)\psi(s,t), \tag{25}$$

and

$$H_B\psi(s,1/t) = s(1-s)\psi(s,1/t). \tag{26}$$

The H_A, H_B are the two partner Hamiltonians in our SUSY-QM model, which is a 1-D model defined on half of the real line: $0 < t < \infty$.

Whereas the hyperbolic plane where the 2-D Laplacian acts, is represented as the upper half of the complex plane given by the coordinates z.

Concluding, the "t" in our $\psi(s,t)$ does correspond to the "z" in $E(s,z)$. Of course, on the Riemann critical line the spectrum $s(1-s)$ is real (and on the real line, trivial zeros). The advantage in our approach is that the inner products of our eigenfunctions $\psi(s,t)$ yield the fundamental function $Z(as+b)$ and there is a one-to-one correspondence between the zeta zeros and the ortogonality conditions on the $\psi(s,t)$ eigenfunctions.

3 The analog of the Riemann hypothesis for the function sin(iz)

It can be proved in an straightforward way that the function of complex variable $\sin(iz)$ has its zeros in the imaginary axis where the real parts of all the zeros are zero $s = 0 + i\pi n$ by simply using the addition law of the sines: $\sin(iz) = \sin(ix - y) = i \sinh x \cos y - \sin y \cosh x = 0 \ \Rightarrow \ x = 0, \ y = \pi n$.

We note that $z = 0 + i0$ is a trivial zero. In this section we will propose a different strategy, based on the symmetry properties of this function.

Our proposal is based on finding the appropriate operator D_1

$$D_1 = -\frac{i}{k}\frac{d}{dx}, \tag{27}$$

such that its eigenvalues s are complex-valued, and its eigenfunctions are given by

$$\psi_s(x) = \frac{1}{2^{1/2}} \, e^{iksx}. \tag{28}$$

We restrict x to be into the interval $[-1, 1]$. Notice that D_1 is not self-adjoint and its eigenvalues are complex valued numbers s.

Thus we have to consider a family of D_1 operators, each characterized by the real number k which can be chosen arbitrarily.

We will only suppose that the following symmetries of our test function $\sin(iz)$ are known,

$$\sin(iz) = \sin(iz + 2i\pi n), \ \sin(-iz) = -\sin(iz), \tag{29}$$

n is an interger.

We define the auxiliary function of the complex variable z, See Figure 1.

$$G(z) = \frac{\sin(iz)}{iz} = G(-z), \tag{30}$$

which is analogous to the Z in the Riemann zeta case [2],

$$Z(z) \equiv \pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\zeta(z). \tag{31}$$

We define an inner product as follows:

$$\langle f|g\rangle = \int_{-1}^{1} f^* g dx. \tag{32}$$

Based on this definition, the inner product of two eigenfunctions of D_1 is,

$$\langle s_1|s_2\rangle \equiv \langle \psi_{s_1}|\psi_{s_2}\rangle = \alpha \int_{-1}^{1} \frac{1}{2} e^{ik(-s_1^* + s_2)x} dx = \frac{\sin[k(-s_1^* + s_2)]}{k(-s_1^* + s_2)}. \tag{33}$$

We note that,

$$\langle s_1|s_2\rangle = G[ik(-s_2+s_1^*)]. \tag{34}$$

Also is easily seen that the inner product of ψ_{s_1} and ψ_{s_2} is equivalent to the inner product of ψ_{s_o} and $\psi_{s_{12}}$, where $s_o = 0+i0$ and $s_{12} = -s_1^*+s_2$. The states will have well defined positive norm.

The inner product (33), expressed in terms of the $\sin(iz)$ function contains the arbitrary parameter k. Using (30) and (33) we obtain:

$$G(iks) = G(-iks). \tag{35}$$

If we replace

$$k \to -k, \tag{36}$$

then the two sides of equation (35) are exchanged, which shows that (36) is a symmetry transformation. We note also that

$$\langle s_1|s_2\rangle = \langle s_2^*|s_1^*\rangle = \langle s_1^*|s_2^*\rangle^*. \tag{37}$$

From (33) we obtain the squared norm of any state ψ_s where $s = x+iy$ is the point (x,y) in the complex plane,

$$\langle s|s\rangle = G(2ky). \tag{38}$$

It has turned out that the norms of all the states having the arguments s with the same real part x are equal, and that all the states localed into the critical line $x=0$ have norm equals to 1.

We will choose the domain of definition of $s = x+iy$ to be inside the critical domain defined by:

$$-1 < x < 1.$$

Here we must to note that the scalar product of ψ_{s_1} and ψ_{s_2} defines the following map S of complex numbers,

$$\begin{aligned} S: \quad \mathcal{C}\otimes\mathcal{C} \quad &\to \quad \mathcal{C} \\ (s_1,s_2) \quad &\mapsto \quad S(s_1,s_2) = G[ik(-s_2+s_1^*)] = G[iks] = G[is']. \end{aligned} \tag{39}$$

Denoting $G[iks_{12}]$ by $G[iks] = G[is'] = \sin(is')/(is')$. The kernel of the map $S(s_1,s_2) = G(ks_{12})$ is given by such values of s that $G(ks_{12}) = 0$. We only need to study the orthogonality (and symmetry) conditions with respect to the "vacuum" state $0+i0$ to prove the SRH. By symmetries of the "orthogonal" states to the "vacuum" we mean always the symmetries of the kernel of the S map. The relationship between s and s' is simply $s' = ks$ which implies that the real and imaginary parts are:

$$x' = kx; \; y' = ky. \tag{40}$$

Let us assume that the putative zeros are located on the vertical lines parallel to the imaginary axis, the critical line for the SRH, which can be written as

$s'_m = x'_m + iy'_{mn}$ where m labels the particular vertical line, and n labels the height of such zero along the vertical line. Hence, for a fixed value of x_m, the value of the real part x'_m can be continuously changed by continuously changing k. And vice versa, x'_m can be held fixed whereas the location of x_m can be continuously changed as one varies k. If we assume that the vertical lines of orthogonal states and zeros belong to a discrete set of lines, instead of a continuum of lines, this requires that $x_m = 0$ is the only consistent value that the orthogonal states can have for their real parts. From this follows that $x'_m = 0$ is the only consistent and possible value which the real part of the zeros of zeta can have. Therefore, the SRH follows directly from the latter conclusion.

However, since the location of the y' values of the zeros varies along the critical line, the imaginary axis, these arguments, of course, cannot provide for the location of the imaginary parts of the zeros. If one has $y' = ky$, it is clear that the fixed points (for all values of k) will be $y = y' = 0$, which is clearly incompatible with the fact that there are no zeros of the function $G(is')$ located in the real horizontal axis and that there are an infinity of zeros of $\sin(is')$ located in the critical line whose imaginary parts are distinct from zero!. To locate both the x and y values of the zeros of $\sin(is)$ we shall follow the SUSY QM model next. Of course one can trivially determine the zeros of $\sin(is)$, but we wish to show now how they can be determined via a SUSY QM model.

4 A fractal supersymmetric quantum mechanical model

The Hilbert-Polya proposal to prove the RH is based on the possibility that the imaginary parts of the nontrivial zeros of zeta are the real eigenvalues of some unknown Hermitian operator [5]. If the nontrivial zeros of the Riemann zeta function are given by $s_n = 1/2 + i\lambda_n$, and if there exists a suitable Hermitian operator $\hat{T}$, whose real eigenvalues are λ_n, then the RH is true. Hence, the zeros s_n are consequently given the complex eigenvalues of the operator $1/2 + i\hat{T}$.

Before constructing the fractal SUSY QM model to prove the RH based on the Hilbert-Polya proposal, let's consider the analogous problem (almost trivial) for the SRH described in Section 3. The SUSY QM model involves two isospectral operators $H^{(+)}$ and $H^{(-)}$ which are defined in terms of the so called SUSY-QM potential $\Phi(x)$. Our ansatz for the SUSY-QM potential associated to the SRH is given by:

$$\Phi(x) = \frac{\pi x}{2}. \tag{41}$$

Note that the SUSY potential is real and it is consistent with the SUSY requirement that $\Phi(x)$ is antisymmetric in x in order to vanish at the origin so that $\Phi^2(x)$ is a symmetric function with a minimum at $x = 0$:

$$\Phi^2(x) = \frac{\pi^2 x^2}{4}. \tag{42}$$

Using such SUSY potential Φ the following SUSY Schrödinger equation associated with the $\hat{H}^{(+)}$ Hamiltonian [33], is:

$$\left(\frac{\partial}{\partial x} + \Phi\right)\left(-\frac{\partial}{\partial x} + \Phi\right)\psi_n^{(+)}(x) = \lambda_n^{(+)}\psi_n^{(+)}(x), \tag{43}$$

where we choose the natural units $\hbar = 2m = 1$. The isospectral condition of the SUSY-QM model requires that $\lambda_n^{(+)} = \lambda_n^{(-)} = \lambda_n$.

The eigenfunction $\psi_n^{(+)}(x)$ associated with the Schrodinger equation for the harmonic oscillator-like potential is the usual Gaussian times a Hermite polynomial and has for corresponding eigenvalues $\lambda_n = \hbar\omega(n + 1/2)$ where the natural frequency is $\omega = (k/m)^{1/2}$.

The potential $V(x)$ of an ordinary QM problem associated with the SUSY-QM model is given by (41):

$$V^{\pm}(x) = \left[\Phi^2(x) \pm \frac{d\Phi(x)}{dx}\right] = \left(\frac{\pi^2}{4}x^2 \pm \frac{\pi}{2}\right). \tag{44}$$

The above potentials $V^{\pm}(x)$ correspond to a harmonic oscillator, whose natural frequency is $\omega = (k/m)^{1/2} = (\pi^2/2m)^{1/2} = \pi$. shifted by an additive postive/negative constant, respectively, and the energy eigenvalues are given by $\lambda_n = \pi(n+1)$ and πn respectively. In order to have the isospectral condition of SUSY QM $\lambda_n^{(+)} = \lambda_n^{(-)} = \lambda_n$ we must have two different values of n, n' such that $n' + 1 = n$. This immediately determines the corresponding eigenfunctions of the two harmonic oscillator partner Hamiltonians.

As we have discussed earlier, the non trivial zeros of the function $\sin(iz)$ are located at $z = 0 + in\pi$, for $n = 0, \pm 1, \pm 2.....$ which is consistent with the equally spaced eigenvalues of the harmonic oscillator QM problem. This means that it is possible to find an ordinary QM Hamiltonian related to a SUSY-QM model and such that their eigenvalues coincide with the imaginary part of the zeros of $\sin(iz)$. This is the Hilbert-Polya implementation to prove the SRH in a nontrivial fashion.

Next, we formulate an inverse eigenvalue problem associated with equation (43), where the λ_n's are to be taken as the imaginary parts of the non trivial zeros of our test function $\sin(is)$. The quantization conditions using the fermionic phase path integral approximation, when aplicable, (the SUSY-QM analog of WKB formula in QM) are based on the CBC formula, the Comtet, Bandrauk and Campbell formula [33]) which reads, after using the natural units $\hbar = 2m$ $= 1$, so that all quantities are suitably written in dimensionless variables for simplicity,

$$I_n(x_n, \lambda_n; a) \equiv 4\int_0^{x_n} dx\,\left[\lambda_n - \Phi^2(x)\right]^{1/2} = 4\int_0^{x_n} dx\,\left[\lambda_n - \pi^2 x^2/4\right]^{1/2} = \pi n, \tag{45}$$

where we take the positive values $n = 1, 2,...$ and the λ_n are the imaginary parts of the nontrivial zeros of $\sin(iz)$. The factor of four in equation (45) orginates because one is integrating over a full cycle. The integration between $0, x_n$ represents a quarter of a cycle. Due to the fact that $\Phi^2(x)$ is an even function of x, in order for supersymmetry to be maintained, the left/right turning points obey are symmetrically located: $x_L^{(n)} = -x_R^{(n)}$ for all orbits, and for each $n =$ 1, 2,... We define $x_n = x_R^{(n)}$.

The second set of equations are provided by the location of the turning points of the bound state orbits and which are defined by:

$$\Phi^2(x = x_n) = \lambda_n = \pi n; \; n = 1, 2, ... \qquad (46)$$

The precise location of the turning points is what is needed in order to evaluate the previous definite integral (the CBC formula) and yield the exact values πn.

The equations (44,45, 46) are the ones we are looking for. The (right) turning points x_n, are defined in terms of all the λ_n, and the well defined CBC formula is the one which involves the zeros λ_n associated with the SUSY potential $\Phi(x)$.

Now let us turn to the fractal SUSY QM problem associated to the Riemann Hypothesis. Armitage [34], considered that the RH can be expressed in terms of diffusion processes with an imaginary time. In this way the Hamiltonian of some QM system could be constructed, which in turn implements the Hilbert-Polya's original program.

A numerical exploration of the Hilbert-Polya idea was recently done by Wu and Sprung [12]. The potential found in [12] has random oscillations around an average value, the average potential allowed them to construct a conventional Hamiltonian whose density of states coincides with the average distribution of the imaginary parts of the Riemann's zeta non trivial zeros. The fluctuations are necessary in order to make the individual eingenvalues fit a set of such zeros within a prescribed error bound. They found that the imaginary parts of the 500 lower lying nontrivial Riemann zeros can be reproduced by a one-dimensional local-potential model, and that a close look at the potential suggests that it has a fractal structure of dimension $D = 1.5$. The references [35, 36, 37] deal with fractal properties of the Riemann zeta function.

One of us [20], was able to consider a p-adic stochastic process having an underlying hidden Parisi-Sourlas supersymmetry, as the effective motion of a particle in a potential which can be expanded in terms of an infinite collection of p-adic harmonic oscillators with fundamental (Wick-rotated imaginary) frequencies $\omega_p = i \ln\ p$ (p is a prime) and whose harmonics are $\omega_{p,n} = i \ln\ p^n$. Here, inspired in a work by Wu and Sprung [12] the p-adic harmonic oscillators are substituted by Weierstrass functions. In this way, we propose a way to construct a Hilbert-Polya operator by using (fractal) SUSY-QM arguments.

In SUSY-QM two isospectral operators $H^{(+)}$ and $H^{(-)}$ are defined in terms of the so called SUSY-QM potential. A SUSY-QM model was proposed in [20] based on the pioneering work of B. Julia [38], where the zeta-function and its fermionic version were related to the partition function of a system of p-adic oscillators in thermal equilibrium at a temperature T. The fermionic zeta-function has zeros at the same positions of the ordinary Riemann function plus a zero at $1/2 + 0i$, this zero is associated to the SUSY ground state. See also the reference [14]:

$$Z_f = \frac{\zeta(s)}{\zeta(2s)} = \sum_n \frac{|\mu(n)|}{n^s}, \tag{47}$$

where $\mu(n)$ is the Mobius function.

Here we consider a fractal potential, defined by a set unknown phases, to be determined after using the CBC formula, associated with a Weierstrass function, continuous but nowhere differentiable. A fractal SUSY-QM Hamiltonian, using fractional derivatives, can be constructed in principe, whose eigenvalues coincide with the imaginary parts of the nontrivial zeros of the zeta, λ_n. The fractal dimension of the potential is $D = 1.5$ and the sought-after phases will be determined by solving the inverse eigenvalue problem via the CBC formula..

Our ansatz for our fractal SUSY-QM potential is based on the Weierstrass fractal function, continuous and nowhere differentiable functions.

$$W(x, \gamma, D, \alpha_n) = \sum_{n=0}^{\infty} \frac{1 - e^{ix\gamma^n}}{\gamma^{n(2-D)}} e^{i\alpha_n}, \tag{48}$$

n are integers, the powers γ^n are the corresponding set of frequencies and the α_n are the sought-after phases. The expansion (48) is convergent if $1 < D < 2$ and $\gamma > 1$. For these values of the parameters the function W is continuous but nowhere differentiable and has D for fractal dimension [39, 40]. One could use for the frequencies suitable powers p^n of a given prime p number, however, we must study the most general case and have powers γ^n for all real values of $\gamma > 1$.

The aim is to relate the SUSY potential-squared Φ^2 to the fractal function $W(x, \gamma, D, \alpha_n)$ defined before. The choice for the $\Phi^2(x)$ expression that appears in the fractal version of the CBC formula will be comprised of a smooth part given by the Wu-Sprung potential $V_{WS}(x)$ plus an oscillatory fluctuating Weierstrass part:

$$\Phi^2(x) = V_{WS}(x) + \frac{1}{2}[W(x, D, \gamma, \alpha_n) + W(-x, D, \gamma, \alpha_n) + c.c] + \phi_o, \tag{49}$$

where we have symmetrized the function $W(x, D, \gamma, \alpha_n)$ with respect to the x variables and taken the real part by adding its corresponding complex conjugate (cc). An additive constant ϕ_o has been included also in order to have a vanishing Φ^2 at the origin $x = 0$. Supersymmetry requires that the Φ^2 is symmetric and vanishes at the origin.

In [12] it was shown that the smooth value of the potential V_{WS} can be obtained as solution of the Abel integral equation. The Wu-Sprung potential $V_{WS}(x)$ is given implicitly as:

$$x = x(V) = \frac{V_o^{1/2}}{\pi} \left[(y-1)^{1/2} \ln \frac{V_o}{2\pi e^2} + y^{1/2} \ln \frac{y^{1/2} + (y-1)^{1/2}}{y^{1/2} - (y-1)^{1/2}} \right]. \tag{50}$$

where the rescaled variable is $y = V/V_o$, and $V_o = 3.10073\,\pi$.

With the SUSY potential Φ at hand one may construct the following SUSY Schrödinger equation associated with the $\hat{H}^{(+)}$ Hamiltonian [33],

$$\left(\mathcal{D}^{(\beta)} + \Phi\right)\left(-\mathcal{D}^{(\beta)} + \Phi\right)\psi_n^{(+)}(x) = \lambda_n^{(+)}\psi_n^{(+)}(x), \tag{51}$$

where we set $\hbar = 2m = 1$. The isospectral condition of the SUSY-QM model requires that $\lambda_n^{(+)} = \lambda_n^{(-)} = \lambda_n$. See in [41] an investigation on fractional Laplacians, and in [42] on vector calculus in fractal domains.

The fractal character of the SUSY QM model suggests that equation (51) is actually an stochastic equation. Instead of the usual derivative d/dx we should use the Riemann-Liouville definition of the fractional derivative, as follows,

$$\mathcal{D}^{(\beta)}F(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_{-\infty}^{t}\frac{F(t')}{(t-t')^{\beta}}dt', \tag{52}$$

where $0 < \beta < 1$. Similarly, the fractional integral of order β is

$$\mathcal{D}^{(-\beta)}F(t) = \frac{1}{\Gamma(\beta)}\int_{-\infty}^{t}\frac{F(t')}{(t-t')^{1-\beta}}dt', \tag{53}$$

where $0 < \beta < 1$. Notice that the lower limits of integration have been chosen to be $-\infty$. In general these choices may vary.

With these ingredients we are prepared to manage the inverse eigenvalue problem associated with equation (51), where the λ_n's are to be taken as the imaginary parts of the non trivial Riemann zeta zeros. The SUSY potential Φ, is related to the ordinary potential through the usual rule $V(x) = \Phi(x)^2 + \mathcal{D}^{(\beta)}\Phi(x)$, where instead of the usual derivative we should use the fractional derivative (52).

We proceed with our ansatz by showing why $\beta = d = D/2$ and $D = 1.5$. This choice is justified based on the fractal dimension of the Wu-Sprung potential of the order of $D = 1.5$ using the first 500 zeros. The reason why $\beta = d = D/2 = 3/4$ is due to the fact that the two terms which define the fractional (fractal) operator $\mathcal{D}^{(\beta)} + \Phi$ in (51) must have the same fractal dimension. If the fractal $\dim(\Phi) = \mathrm{d} = \dim(\mathcal{D}^{(\beta)}) = \beta$, according to the properties of $\mathcal{D}^{(\beta)}$ given in [40], if the fractal $\dim(\Phi) = d$, then $\dim(\mathcal{D}^{(\beta)}\Phi) = \beta + d = 2\beta$. Similarly, for the anti-derivative $\dim(\mathcal{D}^{(-\beta)}\Phi) = -\beta + d$. Hence, one finally has that the fractal dim $(\Phi^2) = 2d = \beta + d = 2\beta = D$. From which one infers that $\beta = D/2 = 3/4$ and it satisfies the required condition for the order β of the fractional derivative, $0 < \beta < 1$.

Therefore, the quantization conditions using the fractal extension of the fermionic phase path integral approximation (the CBC formula) are:

$$I_n(x_n, \lambda_n) \equiv 4\frac{1}{\Gamma(\beta)}\int_0^{x_n} dx' \frac{\left[\lambda_n - \Phi^2(x')\right]^{1/2}}{(x_n - x')^{1-\beta}} = \pi n, \tag{54}$$

where $\beta = D/2 = 3/4$ and $n = 1, 2,...$ and λ_n are the imaginary parts of the nontrivial zeros of zeta. $\Phi^2(x, D)$ is an even function of x so the left/right turning points: $x_L^{(n)} = -x_R^{(n)}$ for all orbits, for each $n = 1, 2,...$ We define $x_n = x_R^{(n)}$.

The second set of equations are given by the definition of the turning points of the bound state orbits:

$$\Phi^2(x_n) = \lambda_n; \; n = 1, 2, ... \tag{55}$$

So, from the three sets of equations (49,54,55) we get what we are looking for, the relationships among the phases, α_n, the (right) turning points x_n, and the imaginary parts of the zeta zeros λ_n.

This is where the determination of the parameter $\gamma > 1$ (the frequencies of the Weierstrass function are γ^n) will come into play. One still has the freedom to vary such parameter at will. This parameter can be fixed through an optimization procedure. One has a one-parameter family of phases α_n which depend on the values λ_n as well as the parameter $\gamma > 1$. One must go back to the orginal fractal SUSY QM wave equation to ensure in fact that the SUSY potential Φ reproduces the orginal λ_n for eigenvalues. The error terms will depend on the different choices of γ. The minimization of the error terms should select, in principle, the optimum choice for $\gamma > 1$ compatible with the SUSY QM wave equation. It would be intriguing to see if $\gamma = 1.618$, the Golden Mean, since the Golden Mean appears in the the theory of Quantum Noise related to the RH [17].

Since Φ^2 is a well defined function, despite that it is not differentiable, it will not affect the fractal extension of the CBC formula because the integrand does not involve its derivatives. Consequently, we can use the definition of fractal anti-derivation (integration) of Rocco-West (53) and write down the fractal-analog of the CBC formula:

$$I_n[x = x_n] - I_n[x = 0] = \pi n; \quad n = 1, 2, 3, ..., \tag{56}$$

where by the I_n's we mean the Rocco-West formula (53) for the fractal integration whose upper limits are $x = x_n$ and $x = 0$ and the lower limits of the Rocco-West formula are $-\infty$. If one wishes one can use the Rocco-West formula with x_n in the upper limit and $x = 0$ in the lower limit. The well defined integrand to be used in the Rocco-West formula is precisely $[1/(x_n - x')^{1-\beta}] \times [\lambda_n - \Phi^2(x')]^{1/2}$ where $\beta = D/2 = 3/4$. This is nothing but the anti-derivative analog of the CBC formula. The turning points are defined as usual, $\Phi^2(x_n) = \lambda_n$ since Φ^2 is a well defined function involving a generalized Weistrass function (for its fluctuating part) and the Wu-Sprung potential (for its smooth part).

By "fractal" SUSY QM model one means a factorization of a Hamiltonian into two products of operators involving fractional derivatives of irrational order. A model of fractal spin has been studied by da Cruz [21]. Our model must not be confused also with those involving fractional supersymmetries in the string literature.

To conclude, we have a well defined extension of the CBC formula based on a fractal SUSY QM model, that gives a direct one-to-one correspondence among the imaginary parts of the zeros λ_n and the phases α_n. This procedure defines the fractal SUSY QM model which yields the imaginary parts of the zeros of zeta implementing the Hilbert-Polya proposal to prove the Riemann Hypothesis. It is warranted to see if the statistical distribution of these phases α_n has any bearing to random matrix theory and the recent studies of quantum phase-locking, entanglement, Ramanujan sums and cyclotomy studied by [43].

The eigenvalue problem for the $H^{(+)}$ Hamiltonian can be reduced to diagonalize an infinite matrix, whose matrix elements can be easily obtained once a convenient basis is found. This matrix involves an infinite set of unknowns in order to have the Riemann's zeros as eigenvalues. A numerical evaluation for each convenient truncation of the matrix is possible. One concludes that the phases of the Weierstrass fractal function appearing in the definition of the $\Phi^2(x)$, namely the square of the SUSY-QM potential, eq. (49), α_n are only approximately found by this method. However this approach has the advantage to give us some clues about the nature and the precise expression of the (square of) SUSY-QM potential. Of course equation (51) could, in principle, be numerically treated following numerical procedures analogous to those used in [12] to give values of the unknown phases α_n within prescribed error bounds.

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Figures

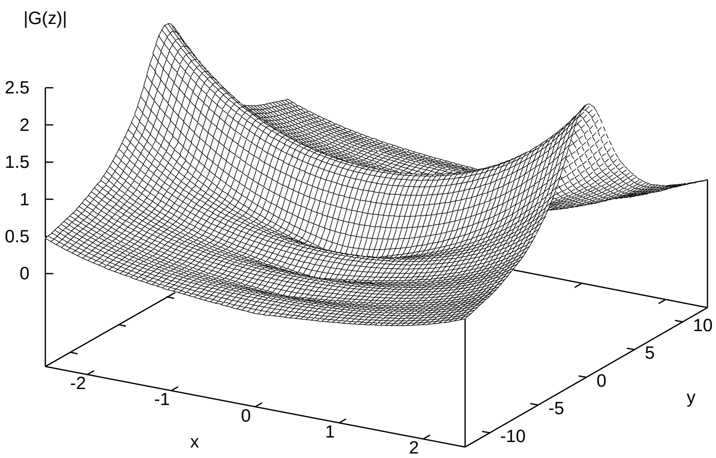

Figure 1: Plot of the absolute value of the function $G(z) = (1/z)\sin(ilz);\ z = x + iy$.

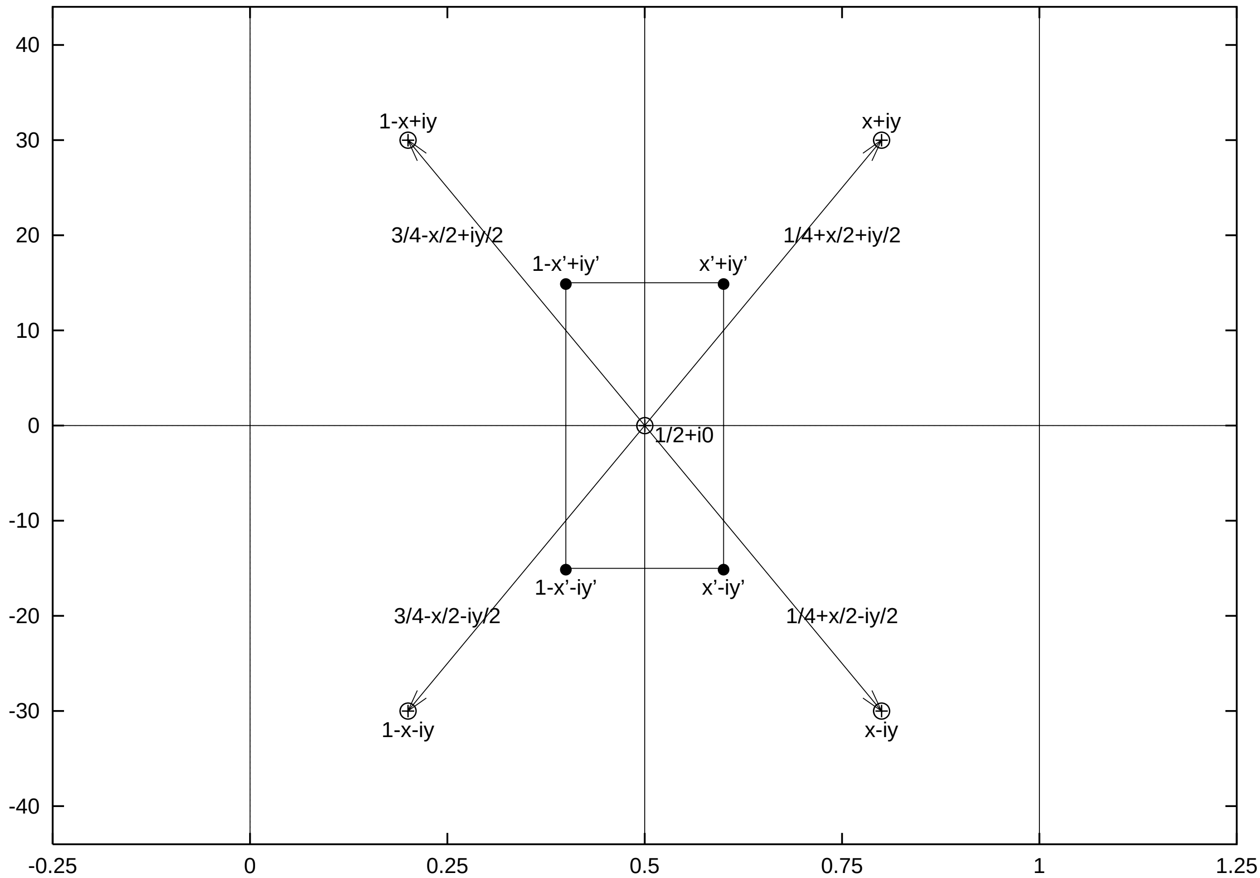

Figure 2: The dots represent generic zeros of the ζ. The crosses represent generic states orthogonal to the reference state $1/2+0i$. The numbers $3/4-x/2-iy/2$, etc, are the arguments of Z appearing in the orthogonality relations between states orthogonal to the reference state. Due to the functional equation of the Riemann zeta-function, these arguments are just the average values between $1/2+0i$ and those orthogonal states. Here we are referring the particular case $k=1$, $l=4$.