Motivic L–functions and regularized determinants

by Christopher Deninger

0. Introduction

In the papers [De1] , [De2] we gave an interpretation of local L–factors of pure motives as regularized characteristic power series on infinite dimensional cohomologies. This lead to speculation on an "arithmetic site" whose global cohomologies would be deeply connected with the global L–series of motives. These arguments suggested in particular a formula for the Riemann ξ–function as a regularized characteristic power series which was proved in [De2] §4.

In sections 1 to 6 of this article we extend the above interpretation to the local L–factors of mixed motives. For the finite primes we give an improved construction of the infinite dimensional cohomologies using an elementary case of the Riemann–Hilbert correspondence. This does away with the semisimplicity assumption we had to make in [De2]. This new point of view was noted independently by S. Bloch. We also understand better than in [De1] the relation between archimedian and Deligne cohomology.

Apart from this our main objective is to discuss in some detail the following aspects of the still speculative "arithmetic cohomology":

What form should a Lefschetz fixed point formula take? We mention the relation with explicit formulas in analytic number theory.

We give a short "proof" in the spirit of [Se] of the Riemann hypotheses assuming that a Hodge $*$–operator with standard properties exists on the prospected cohomologies.

Following a classical pattern we relate the functional equation for motivic L–series to Poincaré duality.

We "explain" the well known conjectures on the vanishing and pole order of L–functions at integers by certain cohomological conjectures.

We point out relations between a Knneth formula and Kurokawa's multiple zeta functions.

In sections 1 to 6 everything is proved and we think of mixed motives in terms of realizations [D5], [J2]. In the speculative §7 we are not precise about the meaning of the word motive in the formal discussions.

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1. Regularized determinants and dimensions

We recall the definition of regularized determinants in the following algebraic setting [De1] §1:

Let Θ be an endomorphism of a complex vector space V of countable dimension. We say that $\det_\infty \Theta$ (or $\dim_\infty \Theta$) is defined if the following conditions 1) and 2) hold.

1) V is the direct sum of finite–dimensional Θ–invariant subspaces. For any α in $\mathbb{C}$ there are at most finitely many of these subspaces on which α occurs as an eigenvalue.

This is equivalent to

1') $\quad V = \bigoplus_{\alpha \in \mathbb{C}} V_\alpha \quad$ where the V_α are Θ–invariant finite dimensional subspaces such that α is the only eigenvalue of $\Theta | V_\alpha$.

If 1') holds then V_α is uniquely determined as $V_\alpha = \mathrm{Ker}\,(\Theta - \alpha)^n$ for n large enough and we call $m(\alpha) := \dim V_\alpha$ the (algebraic) multiplicity of α. We also write $V^{\Theta \sim \alpha}$ for V_α.

2) Under condition 1) let $\mathrm{Sp}\,(\Theta)$ be the set of eigenvalues of Θ with their (algebraic) multiplicities. We assume that the Dirichlet series

$$\sum_{\substack{\alpha \in \mathrm{Sp}\,\Theta \\ \alpha \neq 0}} \frac{1}{\alpha^s} \quad \text{with } \alpha^{-s} = |\alpha|^{-s} e^{-is(\mathrm{Arg}\,\alpha)}\,, \; -\pi < \mathrm{Arg}\,\alpha \leq \pi$$

converges absolutely for $\mathrm{Re}\, s \gg 0$ and has an analytic continuation denoted $\zeta_\Theta(s)$ to the half plane $\mathrm{Re}\, s > -\varepsilon$ for some $\varepsilon > 0$ which is holomorphic at $s = 0$.

Under these conditions we set

$$\dim_\infty(\Theta|V) = \dim V_0 + \zeta_{\tilde{\Theta}}(0)$$

where $\tilde{\Theta}$ is the induced endomorphism of V/V_0 and

$$\det_\infty(\Theta|V) = \begin{cases} \exp(-\zeta'_{\tilde{\Theta}}(0)) & \text{if } \; 0 \notin \mathrm{Sp}\,(\Theta) \\ 0 & \text{if } \; 0 \in \mathrm{Sp}\,(\Theta)\,. \end{cases} \tag{1.1}$$

Remark: The choice of the principal branch Arg of arg is compatible with the convention in [De1] §1 but different from the one in [De2] (2.1). It leads to a more uniform expression for local L–factors in terms of regularized characteristic power series than the one in [De2].

(1.2) Lemma: Consider a commutative diagram with exact lines

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \\ & & \big\downarrow \Theta' & & \big\downarrow \Theta & & \big\downarrow \Theta'' & & \\ 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \end{array}$$

in which $\det_\infty \Theta'$ and $\det_\infty \Theta''$ are defined. Then $\det_\infty \Theta$ is defined as well and

$$\begin{aligned} \det{}_\infty \Theta &= \det{}_\infty \Theta' \cdot \det{}_\infty \Theta'' \\ \dim_\infty \Theta &= \dim_\infty \Theta' + \dim_\infty \Theta''\,. \end{aligned}$$

Proof: By assumption

$$V' = \bigoplus_\alpha V'_\alpha \quad \text{and} \quad V'' = \bigoplus_\alpha V''_\alpha \quad \text{where the}$$

finite dimensional subspaces V'_α, V''_α are given by

$$V'_\alpha = \mathrm{Ker}\,(\Theta' - \alpha)^{n'_\alpha} \quad \text{and} \quad V''_\alpha = \mathrm{Ker}\,(\Theta'' - \alpha)^{n''_\alpha}$$

for n'_α and n''_α large enough.

We claim that for $n_\alpha = n'_\alpha + n''_\alpha$ the natural sequence

$$0 \longrightarrow V'_\alpha \longrightarrow \mathrm{Ker}\,(\Theta - \alpha)^{n_\alpha} \longrightarrow V''_\alpha \longrightarrow 0$$

is exact. For $v'' \in V''_\alpha$ choose a preimage v in V. Then $(\Theta - \alpha)^{n''_\alpha} v \in V'$. Hence

$$(\Theta - \alpha)^{n_\alpha} v = \sum_{\mu \neq \alpha} v'_\mu \quad \text{for certain } v'_\mu \in V'_\mu \; .$$

Since $\Theta' - \alpha$ restricted to V'_μ is an isomorphism for $\mu \neq \alpha$ we find $w'_\mu \in V'_\mu$ with $(\Theta - \alpha)^{n_\alpha} w'_\mu = v'_\mu$. Hence $v - \sum\limits_{\mu \neq \alpha} w'_\mu$ is a preimage of v'' in $\operatorname{Ker}(\Theta - \alpha)^{n_\alpha}$.

Thus $V = \bigoplus V_\alpha$ with the finite dimensional Θ–invariant subspaces $V_\alpha = \operatorname{Ker}(\Theta - \alpha)^{n_\alpha}$ is the decomposition as in condition 1') above. Moreover it follows that the (algebraic) multiplicity of the eigenvalue α on V is the sum of its multiplicities on V' and V''. The remaining assertions are now obvious.

Note that for a positive real number $\delta > 0$ we have

$$\det\nolimits_\infty(\delta\Theta | V) = \delta^{\dim_\infty(\Theta|V)} \det\nolimits_\infty(\Theta | V) \; .$$

In connection with functional equations we will also need the case $\delta = -1$. This will involve regularised superdimensions $\operatorname{sdim}_\infty \Theta$ defined as follows:

(1.3) Assume that for a pair (V, Θ) satisfying condition 1) above we are given a decomposition

$$V = V^+ \oplus V^-$$

into Θ–invariant subspaces. Then $(V^\pm, \Theta^\pm)$ with $\Theta^\pm = \Theta | V^\pm$ also satisfy condition 1). We say that $\operatorname{sdim}_\infty \Theta$ exists with respect to this decomposition, if the Dirichlet series attached to $\Theta^\pm$

$$\sum_{\substack{\alpha \in \operatorname{Sp} \Theta^\pm \\ \alpha \neq 0}} \frac{1}{\alpha^s} \quad \text{with } \alpha^{-s} = |\alpha|^{-s} e^{-is(\operatorname{Arg} \alpha)}$$

converge absolutely for $\operatorname{Re} s \gg 0$ and have analytic continuations $\zeta_{\Theta^\pm}(s)$ to $\operatorname{Re} s > -\varepsilon$ for some $\varepsilon > 0$ with at most first order poles at $s = 0$. Writing:

$$\zeta_{\Theta^\pm}(s) = \frac{\lambda^\pm}{s} + H^\pm(s) \quad , \; \lambda^\pm \text{ in } \mathbb{C} \, , \; H^\pm \text{ holomorphic at } s = 0$$

we set

$$\operatorname{sdim}_\infty \Theta = (\dim V_0^+ + H^+(0)) - (\dim V_0^- + H^-(0)) \; .$$

The reason why we have to allow for poles at $s = 0$ of first order will become clear from the discussion below and from considerations on Hecke L–series (7.19), (7.20).

Remark: If $\dim_\infty \Theta^\pm$ exist, then $\mathrm{sdim}_\infty\Theta$ exists as well and we have:

$$\mathrm{sdim}_\infty\Theta = \dim_\infty \Theta^+ - \dim_\infty \Theta^- \ .$$

In particular if $\dim V < \infty$ we get:

$$\mathrm{sdim}_\infty\Theta = \dim V^+ - \dim V^- \ .$$

We call a decomposition $V = W^+ \oplus W^-$ into Θ–invariant eigenspaces commensurable with (V^+, V^-) if $V^+ \cap W^+$ (resp. $V^- \cap W^-$) is of finite codimension in V^+ and W^+ (resp. in V^- and W^-). Since V decomposes into the generalized eigenspaces V_α we see that commensurability is equivalent to the existence of Θ–invariant decompositions:

$$\begin{aligned} W^+ &= U^+ \oplus F^+ \quad , \quad V^+ = U^+ \oplus E^+ \\ W^- &= U^- \oplus F^- \quad , \quad V^- = U^- \oplus E^- \end{aligned}$$

such that E^+, E^-, F^+, F^- are finite–dimensional and $F^+ \oplus F^- \cong E^+ \oplus E^-$. Hence we get:

(1.4) Lemma: The regularized superdimension of Θ with respect to $V = V^+ \oplus V^-$ exists if and only if it exists with respect to $V = W^+ \oplus W^-$. In case the superdimensions exist their difference is an even integer.

(1.5) For any pair (V, Θ) as above we define a Θ–invariant decomposition by:

$$\begin{aligned} V^+ &= \bigoplus_\alpha V_\alpha \quad \text{where } \alpha = 0 \text{ or } \alpha \neq 0 \text{ and } -\pi < \mathrm{Arg}\,\alpha \leq 0 \\ V^- &= \bigoplus_\alpha V_\alpha \quad \text{where } \alpha \neq 0 \text{ and } 0 < \mathrm{Arg}\,\alpha \leq \pi \ . \end{aligned}$$

A decomposition of V is called standard, if it is commensurable with this one. We say that the regularized superdimension of Θ exists if it exists with respect to one (and hence any) standard decomposition of V. Note the following simple result:

(1.6) Lemma: Given (V, Θ) assume that $\det_\infty \Theta$ and the regularized superdimension of Θ exist. Then $\det_\infty(-\Theta)$ exists as well and we have:

$$\det{}_\infty(-\Theta) = e^{i\pi(\mathrm{sdim}_\infty \Theta)} \det{}_\infty\Theta \ ,$$

where $\mathrm{sdim}_\infty\Theta$ is made up with respect to any standard decomposition of V.

Remark: If $\dim_\infty \Theta^\pm$ exist, then $\mathrm{sdim}_\infty\Theta$ can be viewed as the η-invariant of Θ in the sense of Atiyah–Patodi–Singer. In this case the lemma is well known e.g. [Wi2].

Proof: According to (1.5) we may assume that $\mathrm{sdim}_\infty \Theta$ is formed with respect to the decomposition in (1.6).

Since

$$\mathrm{Arg}\,(-\alpha) = \begin{cases} \mathrm{Arg}\,\alpha + \pi & \text{if} \quad -\pi < \mathrm{Arg}\,\alpha \le 0 \\ \\ \mathrm{Arg}\,\alpha - \pi & \text{if} \quad 0 < \mathrm{Arg}\,\alpha \le \pi \end{cases}$$

we have:

$$\sum_{\substack{\beta \in \mathrm{Sp}\,(-\Theta) \\ \beta \neq 0}} \frac{1}{\beta^s} = e^{-i\pi s} \sum_{\substack{\alpha \in \mathrm{Sp}\,\Theta^+ \\ \alpha \neq 0}} \frac{1}{\alpha^s} + e^{i\pi s} \sum_{\substack{\alpha \in \mathrm{Sp}\,\Theta^- \\ \alpha \neq 0}} \frac{1}{\alpha^s}$$

which converges absolutely for $\mathrm{Re}\, s$ large. Hence the Dirichlet series on the left is analytically continued to $\mathrm{Re}\, s > -\varepsilon$, $\varepsilon > 0$ by the function:

$$\zeta_{-\Theta}(s) = e^{-i\pi s} \zeta_{\Theta^+}(s) + e^{i\pi s} \zeta_{\Theta^-}(s) \; .$$

On the other hand we have

$$\zeta_\Theta(s) = \zeta_{\Theta^+}(s) + \zeta_{\Theta^-}(s) \; .$$

Writing

$$\zeta_{\Theta^\pm}(s) = \frac{\lambda^\pm}{s} + H^\pm(s)$$

as in (1.4) we find $\lambda^+ + \lambda^- = 0$ since $\det_\infty \Theta$ is defined. Thus

$$\zeta_{-\Theta}(s) = \frac{1}{s}(\lambda^+ e^{-i\pi s} + \lambda^- e^{i\pi s}) + e^{-i\pi s} H^+(s) + e^{i\pi s} H^-(s)$$

is holomorphic at $s = 0$ and hence $\det_\infty(-\Theta)$ is defined as well. The formula for $\det_\infty(-\Theta)$ follows from an immediate computation.

2. Regularized determinants and Riemann–Hilbert correspondence on $\mathbb{G}_m$

We recall the Riemann–Hilbert correspondence [D1], [H] in the elementary case where the underlying variety is $\mathbb{G}_m/\mathbb{C}$. Consider a regular singular algebraic differential equation $(M, \bigtriangledown)$ on $\mathbb{G}_m/\mathbb{C}$. Its sheaf of germs of horizontal sections in the analytic topology defines a local system on $\mathbb{C}^*$ and hence a finite dimensional complex representation of $\pi_1(\mathbb{C}^*, 1)$. The resulting tensor functor between regular singular differential equations and representations of $\pi_1(\mathbb{C}^*, 1)$ is an equivalence of tensor categories.

Let us fix a choice of $i = \sqrt{-1}$ and hence an orientation of $\mathbb{C}$. We identify $\mathbb{Z}$ with $\pi_1(\mathbb{C}^*, 1)$ by mapping 1 to the loop $e^{2\pi i t}$, $0 \leq t \leq 1$. Set $\mathbb{L} = \Gamma(\mathbb{G}_m, \mathcal{O}) = \mathbb{C}[z, z^{-1}]$, $\Theta = z\frac{d}{dz}$ and $\triangle = \mathbb{L}[\Theta]$. By D.E.R.S. $(\mathbb{G}_m)$ we denote the category of left $\triangle = \mathbb{L}[\Theta]$–modules D regular singular at $0, \infty$ which are free of finite rank over $\mathbb{L}$. Since $\mathbb{L}$ is principal we obtain an equivalence $\mathbb{H}$ between D.E.R.S. $(\mathbb{G}_m)$ and the category of finite dimensional complex representations of $\mathbb{Z}$.

By construction we have

$$(2.1) \qquad rk_{\mathbb{L}} D = \dim_{\mathbb{C}} \mathbb{H}(D) \; .$$

Explicitely the functor $\mathbb{H}$ is given as follows: Let $e : \mathbb{C} \to \mathbb{C}^*$, $e(\tau) = \exp(2\pi i \tau)$ be the universal covering of $\mathbb{C}^*$. We will view the composition

$$\mathbb{L} \subset \mathcal{O}(\mathbb{C}^*) \overset{e^*}{\hookrightarrow} \mathcal{O}(\mathbb{C})$$

as an inclusion of $\mathbb{C}$–algebras. If we let Θ act on $\mathcal{O}(\mathbb{C})$ by the derivation $\frac{1}{2\pi i}\frac{d}{d\tau}$ then $\mathcal{O}(\mathbb{C})$ becomes a left $\triangle$–module. The group $\mathbb{Z}$ acts $\triangle$–linearly on $\mathcal{O}(\mathbb{C})$ by translations $(\nu^*\varphi)(\tau) = \varphi(\tau + \nu)$ for ν in $\mathbb{Z}$.

Then we have

$$(2.2) \qquad \mathbb{H}(D) = (D \otimes_{\mathbb{L}} \mathcal{O}(\mathbb{C}))^{\Theta = 0}$$

the kernel of $\Theta \widehat{=} \Theta \otimes \mathrm{id} + \mathrm{id} \otimes \Theta$ on $D \otimes_{\mathbb{L}} \mathcal{O}(\mathbb{C})$ with the induced $\mathbb{Z}$–Operation.

Quite generally the hypercohomology of a regular singular algebraic differential equation equals the cohomology of the corresponding local system [D1] ch. II 6.2, 6.3. In our case we deduce the canonical and elementary isomorphisms

$$(2.3) \qquad H^w(\mathfrak{t}, D) \xrightarrow{\sim} H^w(\mathbb{Z}, \mathbb{H}(D))$$

where the one–dimensional real Lie algebra $\mathfrak{t} = \mathbb{R}$ acts on D by mapping t to $t\Theta$. In other words

$$(2.4) \qquad D^{\Theta=0} \xrightarrow{\sim} \mathbb{H}(D)^{\mathbb{Z}} \quad \text{and} \quad D/\Theta D \xrightarrow{\sim} \mathbb{H}(D)_{\mathbb{Z}}.$$

For α in $\mathbb{C}$ let $\mathbb{L}(\alpha)$ denote the $\triangle$–module which as an $\mathbb{L}$–module is $\mathbb{L}$ itself and on which Θ acts by $\Theta_{\mathbb{L}(\alpha)} = \Theta_{\mathbb{L}} - \alpha\,\mathrm{id}$. For λ in $\mathbb{C}^*$ let $\mathbb{C}(\lambda)$ be the $\mathbb{Z}$–Module whose underlying vector space is $\mathbb{C}$ and on which $\nu \in \mathbb{Z}$ acts by multiplication with λ^ν. Then there is a natural isomorphism $\mathbb{H}(\mathbb{L}(\alpha)) \cong \mathbb{C}(e(\alpha))$. The following remark is now trivial.

(2.5) Remark: Every non–zero object of D.E.R.S. $(\mathbb{G}_m)$ is a succesive extension of objects $\mathbb{L}(\alpha)$.

An object D of D.E.R.S. $(\mathbb{G}_m)$ is in particular a $\mathbb{C}$–vector space with an action by Θ. Let $\mathrm{Sp}\,(\Theta)$ denote the set of eigenvalues of Θ on D. Write F for the (inverse of the monodromy) automorphism on $\mathbb{H}(D)$ given by the action of $-1 \in \mathbb{Z}$. With these notations we have:

(2.6) Corollary: As a $\mathbb{C}$–vector space D decomposes into a countable direct sum of finite dimensional Θ–invariant subspaces. Assigning (algebraic) multiplicities to eigenvalues we have

$$\mathrm{Sp}\,(\Theta) = e^{-1}\mathrm{Sp}\,(F)$$

as sets with multiplicities.

Proof: On $\mathbb{L}(\alpha)$ viewed as a $\mathbb{C}$–vector space Θ has eigenvalues $\nu - \alpha$ for $\nu \in \mathbb{Z}$ with multiplicity one. Since the only eigenvalue of F on $\mathbb{C}(e(\alpha))$ is $e(-\alpha)$ we obtain the assertion for $\mathbb{L}(\alpha)$. The general case follows by induction using (2.5) and the proof of (1.3).

Remark: For objects D of D.E.R.S. $(\mathbb{G}_m)$ and representations H of $\mathbb{Z}$ we introduce twists by

$$D(\alpha) = D \otimes_{\mathbb{L}} \mathbb{L}(\alpha) \quad \text{and} \quad H(\lambda) = H \otimes_{\mathbb{C}} \mathbb{C}(\lambda) \quad \text{for } \alpha \in \mathbb{C}\,,\ \lambda \in \mathbb{C}^*\,.$$

Clearly:

$$D(\alpha)^{\Theta=0} = D^{\Theta=\alpha} \quad \text{and} \quad H(\lambda)^{F=\mathrm{id}} = H^{F=\lambda\mathrm{id}}\,.$$

Since $\mathbb{H}$ is a tensor functor we have natural isomorphisms:

$$\mathbb{H}(D(\alpha)) \cong \mathbb{H}(D)(e(\alpha))$$

and hence applying (2.4) to $D(\alpha)$ we get $D^{\Theta=\alpha} \xrightarrow{\sim} \mathbb{H}(D)^{F=e(\alpha)}$. It follows again that α is an eigenvalue of Θ if and only if $e(\alpha)$ is an eigenvalue of F. Moreover their geometric multiplicities are equal.

(2.7) Lemma: For $\gamma \in \mathbb{C}^*$, $z \in \mathbb{C}$ consider the regularized product [De1] §1

$$\prod_{\nu\in\mathbb{Z}} \gamma(z+\nu) := \begin{cases} \exp(-\zeta'_{\gamma,z}(0)) & \text{if } z \notin \mathbb{Z} \\ 0 & \text{if } z \in \mathbb{Z} \end{cases}$$

where $\zeta_{\gamma,z}(s) = \sum_{\nu\in\mathbb{Z}} [\gamma(z+\nu)]^{-s}$ is defined for $\mathrm{Re}\, s > 1$ by taking $-\pi < \mathrm{Arg}\,(\gamma(z+\nu)) \leq \pi$.

Then we have

$$\prod_{\nu\in\mathbb{Z}} \gamma(z+\nu) = \begin{cases} 1 - e^{-2\pi iz} & \text{if} \quad \mathrm{Im}\,\gamma > 0 \text{ or if } \gamma > 0\ ,\ \mathrm{Im}\, z < 0 \\ & \text{or if} \quad \gamma < 0\ ,\ \mathrm{Im}\, z \leq 0 \\ 1 - e^{2\pi iz} & \text{if} \quad \mathrm{Im}\,\gamma < 0 \text{ or if } \gamma > 0\ ,\ \mathrm{Im}\, z \geq 0 \\ & \text{or if} \quad \gamma < 0\ ,\ \mathrm{Im}\, z > 0 \end{cases}$$

Proof: The assertion is trivial if z is an integer. We use the Hurwitz zeta function $\zeta(s,z)$ which is defined for $\mathrm{Re}\, s > 1$, $z \neq 0, -1, -2, \ldots$ by the series

$$\zeta(s,z) = \sum_{\nu=0}^{\infty} \frac{1}{(z+\nu)^s} \quad , \quad -\pi < \mathrm{Arg}\,(z+\nu) \leq \pi$$

with analytic continuation to s in $\mathbb{C} \setminus \{1\}$. It is known that:

$$\zeta(0,z) = \frac{1}{2} - z \quad \text{and} \quad \partial_s\zeta(0,z) = \log\Gamma(z) - \frac{1}{2}\log 2\pi$$

for a suitable branch of $\log\Gamma(z)$.

For a complex number $\gamma \neq 0$ we introduce the functions

$$\zeta_\gamma(s,z) = \sum_{\nu=0}^{\infty} \frac{1}{(\gamma(z+\nu))^s} \quad , \quad -\pi < \arg\gamma(z+\nu) \leq \pi$$

and

$$\begin{aligned} \zeta_\gamma^-(s,z) &= \sum_{\nu=0}^{\infty} \frac{1}{(\gamma(z-\nu))^s} \quad , \quad -\pi < \arg\gamma(z-\nu) \leq \pi \\ &= \zeta_{-\gamma}(s,-z)\ . \end{aligned}$$

If $\gamma \neq 0$ is not a negative real number, then we have

$$\mathrm{Arg}\,(\gamma(z+\nu)) = \mathrm{Arg}\,\gamma + \mathrm{Arg}\,(z+\nu) \quad \text{for almost all } \nu \geq 0$$

since $\lim_{\nu\to\infty} \mathrm{Arg}\,(z+\nu) = 0$ and $-\pi < \mathrm{Arg}\,\gamma < \pi$.

Hence we have

$$\zeta_\gamma(s,z) = \gamma^{-s}\tilde{\zeta}(s,z)$$

where $\tilde{\zeta}(s,z)$ differs from the Hurwitz zeta function only by taking non–principal arguments in the definition of $(z+\nu)^{-s}$ for at most finitely many ν. Therefore we still have

$$\tilde{\zeta}(0,z) = \frac{1}{2} - z \quad \text{and} \quad \exp(-\partial_s\tilde{\zeta}(0,z)) = \left(\frac{1}{\sqrt{2\pi}}\Gamma(z)\right)^{-1}$$

and hence

$$(2.7.1) \qquad \zeta_\gamma(0,z) = \tfrac{1}{2} - z \quad \text{and} \quad \exp(-\partial_s \zeta_\gamma(0,z)) = \gamma^{\frac{1}{2}-z} \left(\tfrac{1}{\sqrt{2\pi}}\Gamma(z)\right)^{-1} .$$

If $\gamma < 0$ we have for almost all $\nu \geq 0$:

$$\operatorname{Arg}(\gamma(z+\nu)) = \begin{cases} \operatorname{Arg}(z+\nu) + \pi & \text{if} \quad \operatorname{Im} z \leq 0 \\ \operatorname{Arg}(z+\nu) - \pi & \text{if} \quad \operatorname{Im} z > 0 \end{cases}$$

and hence by the same argument as before:

$$(2.7.2) \qquad \zeta_\gamma(0,z) = \tfrac{1}{2} - z \quad \text{and}$$

$$\exp(-\partial_s \zeta_\gamma(0,z)) = \begin{cases} |\gamma|^{\frac{1}{2}-z} e^{i\pi\left(\frac{1}{2}-z\right)} \left(\frac{1}{\sqrt{2\pi}}\Gamma(z)\right)^{-1} & \text{if} \quad \operatorname{Im} z \leq 0 \\ |\gamma|^{\frac{1}{2}-z} e^{-i\pi\left(\frac{1}{2}-z\right)} \left(\frac{1}{\sqrt{2\pi}}\Gamma(z)\right)^{-1} & \text{if} \quad \operatorname{Im} z > 0 . \end{cases}$$

We have

$$\zeta_{\gamma,z}(s) = \zeta_\gamma(s,z) + \zeta_\gamma^-(s,z) - (z\gamma)^{-s}$$

and hence

$$\zeta_{\gamma,z}(0) = 0 .$$

The claim now follows from the equation:

$$\prod_{\nu \in \mathbb{Z}} \gamma(z+\nu) = \exp(-\partial_s \zeta_\gamma(0,z)) \exp(-\partial_s \zeta_{-\gamma}(0,-z))(z\gamma)^{-1}$$

using the formula:

$$\frac{1}{z}\left(\frac{1}{\sqrt{2\pi}}\Gamma(z)\right)^{-1}\left(\frac{1}{\sqrt{2\pi}}\Gamma(-z)\right)^{-1} = i(e^{i\pi z} - e^{-i\pi z}) .$$

Remark: The case $\gamma = 1$ of the lemma was pointed out to me by N. Kurokawa in a response to [De2] (2.3).

From (2.6), (2.7) and (1.7) we draw the following corollary:

(2.8) Corollary: Fix real numbers $\delta > 0$ and $q > 1$. For any D in D.E.R.S ($\mathbb{G}_m$) set $\Theta_q = \frac{2\pi i}{\log q}\Theta$ acting on D. Then we have for complex s:

$$(2.9) \qquad \det{}_\infty(\delta(s \pm \Theta_q)|D) = \det(1 - q^{-s}F^{\mp 1}|\mathbb{H}(D)).$$

A standard decomposition $D = D^+ \oplus D^-$ of D with respect to the operator $-\Theta_q$ is also standard for any operator $\delta(s - \Theta_q)$ with s in $\mathbb{C}$ and we have

$$
(2.10) \qquad \det_\infty(-\delta(s-\Theta_q)|D) = \varepsilon(s)\det_\infty(\delta(s-\Theta_q)|D)
$$

where

$$
\begin{aligned}
\varepsilon(s) &= \exp i\pi(\dim_\infty(\delta(s-\Theta_q)|D^+) - \dim_\infty(\delta(s-\Theta_q)|D^-)) \\
&= (-q^s)^{\dim \mathbb{H}(D)} \det(F|\mathbb{H}(D))^{-1} .
\end{aligned}
$$

Proof. For any $\lambda \neq 0$ let τ_λ be a complex number with $e(\tau_\lambda) = \lambda$. According to (2.6) the eigenvalues of $\delta(s \pm \Theta_q)$ are the numbers

$$
\frac{2\pi i \delta}{\log q}\left(\left(\frac{s\log q}{2\pi i} \pm \tau_\lambda\right) + \nu\right) \quad \text{for } \lambda \in \mathrm{Sp}\,(F)\,,\ \nu \in \mathbb{Z}
$$

with the appropriate multiplicities. Now the first formula follows from (2.7).

It is clear that a decomposition $D = D^+ \oplus D^-$ is standard for $-\Theta_q$ if and only if it is standard with respect to $\delta(s-\Theta_q)$ for any value of s. Thus (1.7) implies formula (2.10) for

$$
\varepsilon(s) = \exp i\pi(\dim_\infty(\delta(s-\Theta_q)|D^+) - \dim_\infty(\delta(s-\Theta_q)|D^-))
$$

taking into account that the regularized dimensions exist by the following argument. Let D^+ (resp. D^-) be the direct sum of the generalized eigenspaces of the operator $-\Theta_q$ for the eigenvalues $\frac{2\pi i}{\log q}(-\tau_\lambda + \nu)$ for $\nu \leq 0$ (resp. $\nu > 0$) and $\lambda \in \mathrm{Sp}\,(F)$. Then the decomposition $D = D^+ \oplus D^-$ is standard for all operators $\delta(s-\Theta_q)$ and using (2.7.1) and (2.7.2) we have for **all** complex s:

$$
\dim_\infty(\delta(s-\Theta_q)|D^+) = \sum_{\lambda\in\mathrm{Sp}\,(F)} \left(\frac{1}{2} + \frac{s\log q}{2\pi i} - \tau_\lambda\right) .
$$

and

$$
\dim_\infty(\delta(s-\Theta_q)|D^-) = \sum_{\lambda\in\mathrm{Sp}\,(F)} \left(-\frac{1}{2} - \frac{s\log q}{2\pi i} + \tau_\lambda\right) .
$$

Hence

$$
\begin{aligned}
&\exp \pi i(\dim_\infty(\delta(s-\Theta_q)|D^+) - \dim_\infty(\delta(s-\Theta_q)|D^-)) \\
&= \prod_{\lambda\in\mathrm{Sp}\,(F)} \exp \pi i \left(1 + \tfrac{s\log q}{\pi i} - 2\tau_\lambda\right) \\
&= (-q^s)^{\dim \mathbb{H}(D)} \det(F|\mathbb{H}(D))^{-1} .
\end{aligned}
$$

Remark: The following particular case of formula (2.9)

$$
\det_\infty(i\Theta|D) = \det(1 - F^{-1}|\mathbb{H}(D))
$$

is closely related with lemma 2 in [A]. Note that F^{-1} is the usual monodromy operator.

(2.11) For $q > 0$ let $\mathcal{D}_q$ be the category D.E.R.S. $(\mathbb{G}_m)$ but with the following notion of twist: Any object D of $\mathcal{D}_q$ is viewed as a representation of the one–dimensional real Lie algebra $\mathfrak{t} = \mathbb{R}$ by mapping 1 to $\Theta_q = \frac{2\pi i}{\log q}\Theta$. For α in $\mathbb{C}$ the twist $D(\alpha)$ of D in $\mathcal{D}_q$ is defined to be D itself as an $\mathbb{L}$–module but with $\mathfrak{t}$–action given by

$$\Theta_{D(\alpha),q} = \Theta_{D,q} - \alpha \,\mathrm{id} \quad \text{i.e. } \Theta_{D(\alpha)} = \Theta_D - \frac{\alpha \log q}{2\pi i}\mathrm{id} \; .$$

We have natural isomorphisms

$$\mathbb{H}(D(\alpha)) \cong \mathbb{H}(D)(q^\alpha) \; .$$

(2.12) Since $\mathbb{H}$ is an equivalence of categories, we can choose a quasi–inverse functor $\mathbb{D}$. For the sequel it will not be important which quasi–inverse we take. None the less we mention the following canonical choice to explain the relation with the construction in [De2].

Let $B = \mathbb{C}[\mathbb{C}]$ be the group algebra over $\mathbb{C}$ with coefficients in $\mathbb{C}$. The typical element will be written in the form $\sum r_\alpha e^\alpha$ with α, r_α in $\mathbb{C}$ and a *symbol* e^α obeying the rule $e^{\alpha+\alpha'} = e^\alpha e^{\alpha'}$. We can view B as a subalgebra of $\mathcal{O}(\mathbb{C})$ by mapping e^α to the function $\tau \mapsto \exp(\alpha\tau)$. Then B inherits a $\mathbb{Z}$ and a $\triangle$–action from $\mathcal{O}(\mathbb{C})$:

$$\nu^*(\textstyle\sum r_\alpha e^\alpha) = \sum r_\alpha \exp(\alpha\nu) e^\alpha \quad \text{for } \nu \text{ in } \mathbb{Z}$$
$$\mathbb{L} = B^{\mathbb{Z}} \; , \; \Theta_B(\textstyle\sum r_\alpha e^\alpha) = \sum (2\pi i)^{-1} \alpha r_\alpha e^\alpha \; .$$

For a finite dimensional complex representation H of $\mathbb{Z}$ let F be the automorphism corresponding to the action of -1. Decompose F into its semisimple and unipotent parts $F = F_s F_u$. Let H_s be the representation of $\mathbb{Z}$ on H where -1 acts by F_s. We set

$$\mathbb{D}(H) = (H_s \otimes_{\mathbb{C}} B)^{\mathbb{Z}} \; .$$

This is naturally an $\mathbb{L}$–module and it becomes a $\triangle$–module by letting Θ act by

$$\frac{1}{2\pi i} \log F_u \otimes \mathrm{id}_B + \mathrm{id}_{H_s} \otimes \Theta_B \; .$$

Decomposing H into eigenspaces of F_s and noting that the λ–eigenspace of $(-1)^*$ on B is isomorphic to $\mathbb{L}(\tau_\lambda)$ for any τ_λ such that $e(\tau_\lambda) = \lambda$ we see that $\mathbb{D}(H)$ is an object of D.E.R.S. $(\mathbb{G}_m)$.

To see that $\mathbb{D}$ is a quasi–inverse to $\mathbb{H}$ we proceed as follows. The map:

$$\begin{array}{rcl} H & \longrightarrow & (H_s \otimes \mathcal{O}(\mathbb{C}))^{\Theta=0} \\ h & \longmapsto & F_u^{-\tau}(h) = \exp(-\tau \log F_u)(h) = \sum_{\nu=0}^{\infty} (-\log F_u)^{\nu}(h) \otimes \frac{\tau^{\nu}}{\nu!} \end{array}$$

is an isomorphism of $\mathbb{C}$–vector spaces by the theory of ordinary differential equations. It is also $\mathbb{Z}$–equivariant:

$$\begin{aligned} (-1)^* F_u^{-\tau}(h) &= F_u^{-(\tau-1)}(F_s h) = F_u^{-\tau}(F_u F_s h) \\ &= F_u^{-\tau}((-1)^* h) \qquad \text{for } h \text{ in } H \ . \end{aligned}$$

Hence we obtain a natural transformation: $\mathbb{H}\mathbb{D} \to \mathrm{id}$ defined by the commutative diagram:

$$\begin{array}{ccc} \mathbb{H}\mathbb{D}(H) = ((H_s \otimes_{\mathbb{C}} B)^{\mathbb{Z}} \otimes_{\mathbb{L}} \mathcal{O}(\mathbb{C}))^{\Theta=0} & \to & (H_s \otimes_{\mathbb{C}} B \otimes_{\mathbb{L}} \mathcal{O}(\mathbb{C}))^{\Theta=0} \\ \downarrow & & \downarrow \mathrm{id} \otimes \text{multipl.} \\ H & \xrightarrow{\sim} & (H_s \otimes_{\mathbb{C}} \mathcal{O}(\mathbb{C}))^{\Theta=0} \ . \end{array}$$

One checks that it induces isomorphisms $\mathbb{H}\mathbb{D}(\mathbb{C}(\lambda)) \xrightarrow{\sim} \mathbb{C}(\lambda)$ for all λ in $\mathbb{C}^*$ Since $\mathbb{H}$ and $\mathbb{D}$ are exact and since every $\mathbb{Z}$–representation is a successive extension of $\mathbb{C}(\lambda)$'s we find that $\mathbb{H}\mathbb{D} \to \mathrm{id}$ is an isomorphism of functors. Hence $\mathbb{H}$ is a quasi–inverse of $\mathbb{D}$.

Remark: In [De2] we took the derivation $\mathrm{id}_{H_s} \otimes \Theta_B$ on $\mathbb{D}(H)$ which is the right one only if F is semisimple. Thus we had to assume certain (conjectured) semi–simplicity properties of the Frobenius action on l–adic cohomology in §2 of loc. cit. These assumptions can now be discarded.

3. The non–archimedean local L–factors

Using the results of section 2 we rewrite the local non–archimedean L–factors of a motive in terms of regularized characteristic power series.

Let K be a local non–archimedean field with prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$ inertia group I and geometric Frobenius automorphism F in $\mathrm{Gal}\,(\overline{\kappa}/\kappa)$ where $\kappa = \mathcal{O}_K/\mathfrak{p}$. Fix a prime number l different from the residue characteristic of K and an embedding $\iota : \mathbb{Q}_l \hookrightarrow \mathbb{C}$. Now assume that char$K = 0$. For a finite extension $E/\mathbb{Q}$ let $\mathcal{MM}_K(E)$ be the category of mixed motives over K in the sense of [D5] or [J2] with multiplication by E. If $M_l = H^\bullet(M \otimes_K \overline{K}, \mathbb{Q}_l)$ denotes the l–adic realization of M we obtain a functor

$$(3.1) \qquad M \longmapsto M_l^I \otimes_{\mathbb{Q}_l,\iota} \mathbb{C} = M_{l,\iota}^I$$

from $\mathcal{MM}_K(E)$ into the category of $(E \otimes \mathbb{C})[F]$–modules of finite rank over $E \otimes \mathbb{C}$. It is expected that these functors for different l and ι are isomorphic [T] (4.2.4). Since $E \otimes \mathbb{C} = \mathbb{C}^{\mathrm{Hom}(E,\mathbb{C})}$ we may view $M_{l,\iota}^I$ as an array of complex vector spaces

$$M_{l,\iota}^I = (M_{l,\iota,\sigma}^I)_{\sigma \in \mathrm{Hom}(E,\mathbb{C})} \text{ where } M_{l,\iota,\sigma}^I = M_{l,\iota}^I \otimes_{E \otimes \mathbb{C},\sigma} \mathbb{C} \; .$$

The $E \otimes \mathbb{C}$–valued local L–factor of M is defined by

$$L_K(M,s) = \left(\det_{\mathbb{C}}(1 - FN\mathfrak{p}^{-s} \mid M_{l,\iota,\sigma}^I)^{-1}\right)_{\sigma \in \mathrm{Hom}(E,\mathbb{C})} \; .$$

A priori it depends on the choice of l and ι although by the above remark it is expected to be independent of this choice. In certain cases this independence is known [T] (4.3.1).

For any $\mathbb{Q}$–linear category $\mathcal{A}$ let $\mathcal{A}(E)$ denote the E–linear category of objects A in $\mathcal{A}$ with multiplication by E:

$$E \longrightarrow \mathrm{End}\, A \; , \; 1 \longmapsto \mathrm{id}_A$$

and with the evident morphisms.

Composing the functor (3.1) with a quasi–inverse $\mathbb{D}$ to the Riemann–Hilbert correspondence $\mathbb{H}$ on $\mathbb{G}_m/\mathbb{C}$ we obtain an E–linear functor:

$$(3.1.1) \qquad \mathcal{F} = \mathcal{F}_{l,\iota} : \mathcal{MM}_K(E) \longrightarrow \text{D.E.R.S.}\ (\mathbb{G}_m)(E) \; .$$

Note that via $\mathcal{F}^\bullet(M) := \mathcal{F}(H^\bullet(M))$ the functor $\mathcal{F}$ is naturally $\mathbb{Z}$–graded. We equip $\mathcal{F}(M)$ with a Lie–algebra ind–representation of $\mathfrak{t}$ by sending t to $t\Theta_{N\mathfrak{p}}$. We will view $\mathcal{F}$ as a functor to $\mathcal{D}_{N\mathfrak{p}}(E)$. Up to natural isomorphisms it commutes with twists by integers.

Let $\mathcal{MM}_K^{\mathrm{good}}(E)$ denote the full subcategory of motives M with good reduction in the sense that $M_l^I = M_l$. The restriction $H^\bullet(\_/\mathbb{L})$ of $\mathcal{F}^\bullet$ to $\mathcal{MM}_K^{\mathrm{good}}(E)$ is a tensor functor between Tannakian categories. In particular $H^\bullet(\_/\mathbb{L})$ can be viewed as a cohomology theory on motives in the sense of Grothendieck.

Let us write $H^w(X/\mathbb{L}) = H(H^w(X)/\mathbb{L})$ for smooth projective varieties X over K such that $H^w(X)$ has good reduction ($E = \mathbb{Q}$). For this cohomology theory $H^w(\ /\mathbb{L})$ we have Poincaré–duality, a Knneth–formula, Chern–classes etc. The definition of $H^w(\_/\mathbb{L})$ via l–adic cohomology is of course not satisfactory. It just shows that such a theory exists. An independent construction would be of great interest. It would also be important to know what the groups $H^w(\_/\mathbb{L})$ should be in the bad reduction case. There we have only constructed the analogue of $H^w(X_{\overline{K}}, \mathbb{Q}_l)^I$.

Up to now we have kept $\mathbb{L}$ fixed and chosen $\Theta_{N\mathfrak{p}}$ which depends on the field K as our preferred derivation. However we can also keep the derivation fixed and vary the spaces as follows:

For $q > 1$ consider the subring

$$\mathbb{L}_q = \mathbb{C}\left[\exp\left(\frac{2\pi i}{\log q}\xi\right)\ ,\ \exp\left(-\frac{2\pi i}{\log q}\xi\right)\right] \subset \mathcal{O}(\mathbb{C})$$

equipped with the derivation $\Theta = \frac{d}{d\xi}$ and set $\mathbb{L}_\mathfrak{p} = \mathbb{L}_{N\mathfrak{p}}$. By the change of variable $z = \exp\left(\frac{2\pi i}{\log N\mathfrak{p}}\xi\right)$ we can identify the pair $(\mathbb{L}, \Theta_{N\mathfrak{p}})$ with $(\mathbb{L}_\mathfrak{p}, \Theta)$. As $(\mathbb{L}_\mathfrak{p}, \Theta)$–module we write $(\mathcal{F}_\mathfrak{p}, \Theta)$ for the pair $(\mathcal{F}, \Theta_{N\mathfrak{p}})$.

Let us write $\mathcal{D}_\mathfrak{p}$ for the category with twists $\mathcal{D}_{N\mathfrak{p}}$ of (2.11) if we make the identification $(\mathbb{L}, \Theta_{N\mathfrak{p}}) = (\mathbb{L}_\mathfrak{p}, \Theta)$ in its construction. Thus we view $\mathcal{F}$ and $H^\bullet(\_/\mathbb{L})$ as functors:

(3.1.1)
$$\mathcal{F}_\mathfrak{p} = \mathcal{F}_{\mathfrak{p},l,\iota} : \mathcal{MM}_K(E) \longrightarrow \mathcal{D}_\mathfrak{p}(E)$$

and

$$H^\bullet(\_/\mathbb{L}_\mathfrak{p}) : \mathcal{MM}_K^{\text{good}}(E) \longrightarrow \mathcal{D}_\mathfrak{p}(E)\ .$$

Note that there is an isomorphism of categories

$$\mathcal{D}_\mathfrak{p}(E) \cong \prod_{\sigma \in \mathrm{Hom}(E,\mathbb{C})} \mathcal{D}_\mathfrak{p} \quad , \quad D \mapsto (D_\sigma)$$

where $D_\sigma = D \otimes_{E \otimes \mathbb{C}, \sigma} \mathbb{C}$.

(3.2) Proposition: For M in $\mathcal{MM}_K(E)$, s in $\mathbb{C}$, $\delta > 0$ we have:

$$L_K(M, s) = \det{}_\infty(\delta(s - \Theta) | \mathcal{F}_\mathfrak{p}(M)_\sigma)^{-1}_{\sigma \in \mathrm{Hom}(E,\mathbb{C})}$$

and in particular for a smooth projective variety X/K with $H^w(X)$ of good reduction:

$$L_K(H^w(X), s) = \det{}_\infty(\delta(s - \Theta) | H^w(X/\mathbb{L}_\mathfrak{p})_\sigma)^{-1}_{\sigma \in \mathrm{Hom}(E,\mathbb{C})}\ .$$

Here it is understood that the same pair l, ι is used for the definition of the local factor and for the definition of $\mathcal{F}_{\mathfrak{p}} = \mathcal{F}_{\mathfrak{p},l,\iota}$. The proposition is then an immediate consequence of the definitions and (2.8).

(3.3) If K is a local field of characteristic $p > 0$ we do not yet have a useful category of motives available. However for any local field we can still define a functor $\mathcal{F}^\bullet$ from the category of smooth projective varieties X/K to D.E.R.S. $(\mathbb{G}_m)$ by setting

$$\mathcal{F}^\bullet(X) = \mathbb{D}(H^\bullet(X_{\overline{K}}, \mathbb{Q}_l)^I \otimes_{\mathbb{Q}_l,\iota} \mathbb{C}) \; .$$

We equip $\mathcal{F}^\bullet(X)$ with a Lie–algebra action of $\mathfrak{t}$ by sending t to $t\Theta_{N\mathfrak{p}}$ and view $\mathcal{F}^\bullet(X)$ as a graded object of $\mathcal{D}_{N\mathfrak{p}}$. If $\mathrm{char}(K) = 0$ we have $\mathcal{F}^\bullet(X) = \mathcal{F}(H^\bullet(X))$ in our earlier notation.

4. Interlude: Varieties over finite fields

In this short section we look at the Weil conjectures from the point of view of a D.E.R.S. $(\mathbb{G}_m)$–valued cohomology theory.

For a variety X over $\mathbb{F}_q$ we set

$$H^\bullet(X/\mathbb{L}) := \mathbb{D}(H^\bullet(\overline{X}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l,\iota} \mathbb{C}) \; , \; \overline{X} = X \otimes \overline{\mathbb{F}}_q$$

where l–adic cohomology is equipped with the geometric Frobenius $F = Fr_q^* = (\mathrm{id} \times (\;)^q)^{*-1}$ and $H^\bullet(X/\mathbb{L})$ is viewed as an object of $\mathcal{D}_q$. Since $\mathbb{D}$ is an exact tensor functor the theory $H^\bullet(\_/\mathbb{L})$ inherits all the usual properties of a cohomology theory. Let

$$H^w(X/\mathbb{L}) = H^w(X/\mathbb{L})^+ \oplus H^w(X/\mathbb{L})^-$$

be a standard decomposition of $H^w(X/\mathbb{L})$ with respect to the operator $-\Theta_q$. Corollary (2.8) implies:

(4.1) Proposition: For any variety $X/\mathbb{F}_q$, $w \geq 0$ and s in $\mathbb{C}$ we have

$$\iota\mathrm{det}_{\mathbb{Q}_l}(1 - q^{-s} Fr_q^* | H^w(\overline{X}, \mathbb{Q}_l)) = \mathrm{det}_\infty(s - \Theta_q | H^w(X/\mathbb{L}))$$

and

$$\mathrm{det}_\infty(-s + \Theta_q | H^w(X/\mathbb{L})) = \varepsilon_w(s) \mathrm{det}_\infty(s - \Theta_q | H^w(X/\mathbb{L}))$$

where

$$\begin{aligned}
\varepsilon_w(s) &= \exp i\pi(\dim_\infty((s-\Theta_q)|H^w(X/\mathbb{L})^+) - \dim_\infty((s-\Theta_q)|H^w(X/\mathbb{L})^-)) \\
&= (-q^s)^{b_w}\iota\det(Fr_q^*|H^w(\overline{X},\mathbb{Q}_l))^{-1} \quad \text{where } b_w = \dim_{\mathbb{Q}_l} H^w(\overline{X},\mathbb{Q}_l)\ .
\end{aligned}$$

If the variety $X/\mathbb{F}_q$ is smooth and proper, Poincaré duality gives a perfect pairing of objects in $\mathcal{D}_q$:

$$H^w(X/\mathbb{L}) \times H^{2d-w}(X/\mathbb{L}) \longrightarrow H^{2d}(X/\mathbb{L}) \xrightarrow[\sim]{Tr} \mathbb{L}(-d) \quad \text{(note (2.11))}$$

where $d = \dim X$. In particular Θ_q on $H^w(X/\mathbb{L})$ has the same eigenvalues with the same (algebraic) multiplicities as $d\cdot \mathrm{id} - \Theta_q$ on $H^{2d-w}(X/\mathbb{L})$. Hence:

$$\begin{aligned}
\det{}_\infty(s-\Theta_q|H^w(X/\mathbb{L})) &= \varepsilon_w(s)^{-1}\det{}_\infty(-s+\Theta_q|H^w(X/\mathbb{L})) \\
&= \varepsilon_w(s)^{-1}\det{}_\infty((d-s)-\Theta_q|H^{2d-w}(X/\mathbb{L}))\ .
\end{aligned}$$

This implies the functional equation for $\zeta_X(s)$ in the form

$$\zeta_X(s) = e^{i\pi(\chi^+(s)-\chi^-(s))}\zeta_X(d-s)$$

where $\chi^\pm(s) = \sum_{w=0}^{2d}(-1)^w\dim_\infty(s-\Theta_q|H^w(X/\mathbb{L})^\pm)$. Note that by Deligne's theorem the eigenvalues of Θ_q on $H^w(X/\mathbb{L})$ have weight w i.e. real part $=\frac{w}{2}$.

5. Logarithmic connections and filtered vector spaces

In this section we relate certain algebraic vector bundles on $\mathbb{G}_{a,\mathbb{C}}$ together with a logarithmic connection to filtered vector spaces. This will be used in section 6 to construct the analogs for archimedian $\mathfrak{p}$ of the functors $\mathcal{F}_\mathfrak{p}$ introduced in section 3.

Let D.E.L.S. $(\mathbb{G}_{a,\mathbb{C}})$ be the category of algebraic vector bundles $\mathcal{V}$ on $\mathbb{G}_{a,\mathbb{C}}$ together with a connection

$$\nabla : \mathcal{V} \longrightarrow \mathcal{V} \otimes \Omega^1_{\mathbb{G}_{a,\mathbb{C}}/\mathbb{G}_{m,\mathbb{C}}}\langle 0\rangle$$

having at most a logarithmic pole at zero.

Set $\mathbb{L}^+ = \Gamma(\mathbb{G}_{a,\mathbb{C}}, \mathcal{O}) = \mathbb{C}[z]$ and $\triangle^+ = \mathbb{L}^+[\Theta]$ where $\Theta = z\frac{d}{dz}$. The category D.E.L.S. $(\mathbb{G}_{a,\mathbb{C}})$ is canonically equivalent to the category of $\triangle^+$–modules D^+ which are free of finite rank over $\mathbb{L}^+$.

For $K = \mathbb{C}$ or $\mathbb{R}$ set $e_K = [\mathbb{C} : K]$ and $\Theta_K = -e_K\Theta$. Let $\mathcal{D}_K$ be the category D.E.L.S. $(\mathbb{G}_{a,\mathbb{C}})$ equipped with the following notion of twist: Any object of $\mathcal{D}_K$ is viewed as a representation of $\mathfrak{t}$ by mapping 1 to Θ_K. For α in $\mathbb{C}$ the twist $D^+(\alpha)$ of D^+ in $\mathcal{D}_K$ is defined to be D^+ itself as an $\mathbb{L}^+$–module but with $\mathfrak{t}$–action given by

$$\Theta_{D^+(\alpha),K} = \Theta_{D^+,K} - \alpha \,\mathrm{id} \quad \text{i.e. } \Theta_{D^+(\alpha)} = \Theta_{D^+} + \frac{\alpha}{e_K}\,\mathrm{id}\ .$$

We need the following categories $\mathcal{F}il_K$:

$\mathcal{F}il_{\mathbb{C}}$ is the additive category of finite dimensional complex vector spaces V with a decreasing filtration $\mathrm{Fil}^r V$ such that $\mathrm{Fil}^{r_1}V = 0$, $\mathrm{Fil}^{r_2}V = V$ for some r_1, r_2. Morphisms are supposed to respect the filtrations.

$\mathcal{F}il_{\mathbb{R}}$ is the additive category of finite dimensional complex vector spaces with a filtration as above and with an involution F_∞ which respects the filtration and induces multiplication by $(-1)^\bullet$ on the associated graded vector space $\mathrm{Gr}^\bullet V$. Morphisms are supposed to respect the filtration and to commute with F_∞.

The categories $\mathcal{F}il_K$ are obviously pseudo abelian i.e. additive and such that kernels and images of projectors exist.

In $\mathcal{F}il_K$ the twist $V(n)$ of an object V by an integer n is defined to be V itself as a vector space but with filtration

$$\mathrm{Fil}^r V(n) = \mathrm{Fil}^{r+n} V$$

and in case $K = \mathbb{R}$ with $F_\infty | V(n) = (-1)^n F_\infty | V$. Write $\mathbb{C}(0)$ for the object in $\mathcal{F}il_K$ whose underlying vector space is $\mathbb{C}$ with filtration given by:

$$\mathrm{Fil}^r\mathbb{C}(0) = \mathbb{C}(0) \quad \text{for} \quad r \le 0 \quad \text{and} \quad \mathrm{Fil}^r\mathbb{C}(0) = 0 \quad \text{for } r > 0\ .$$

In case $K = \mathbb{R}$ we set $F_\infty = \mathrm{id}$ on $\mathbb{C}(0)$. Note that

$$\mathrm{Hom}_{\mathcal{F}il_K}(\mathbb{C}(n), \mathbb{C}(m)) = \begin{cases} \mathbb{C} & \text{for } m \le n\ ,\ m \equiv n \bmod e_K \\ 0 & \text{otherwise} \end{cases}\ .$$

On $\mathbb{L} = \Gamma(\mathbb{G}_{m,\mathbb{C}}, \mathcal{O})$ consider the filtration and involution given by:

$$\mathrm{Fil}^r \mathbb{L} = z^r \mathbb{L}^+ \quad , \quad F_\infty(z) = -z \; .$$

For V in $\mathcal{F}il_\mathbb{C}$ set

$$\mathbb{D}^+(V) = \mathrm{Fil}^0(V \otimes_\mathbb{C} \mathbb{L})$$

an $\mathbb{L}^+ = \mathrm{Fil}^0\mathbb{L}$–module with action by $\Theta_\mathbb{C} \mathrel{\widehat{=}} \mathrm{id} \otimes \Theta_\mathbb{C}$.

For V in $\mathcal{F}il_\mathbb{R}$ set

$$\mathbb{D}^+(V) = \mathrm{Fil}^0(V \otimes_\mathbb{C} \mathbb{L})^{F_\infty = \mathrm{id}} \; .$$

Let sq be the injection $\mathbb{L} \to \mathbb{L}$ induced by $sq(z) = z^2$. Note that it corresponds to the squaring map on $\mathbb{G}_{m,\mathbb{C}}$. We view $\mathbb{D}^+(V)$ as an $\mathbb{L}^+$–module via sq and let $\Theta_\mathbb{R}$ operate via $\mathrm{id} \otimes \Theta_\mathbb{C}$.

For $K = \mathbb{R}, \mathbb{C}$ there are natural isomorphisms $\mathbb{D}^+(\mathbb{C}(n)) \cong \mathbb{L}^+(n)$ in $\mathcal{D}_K$.

Similarly as in [De1] §6 we have the following easy proposition:

(5.1) Proposition: In $\mathcal{F}il_K$ any object is isomorphic to a finite direct sum of objects $\mathbb{C}(n)$ for n in $\mathbb{Z}$. The $\mathbb{D}^+$ as constructed above induces an additive functor

$$\mathbb{D}^+ : \mathcal{F}il_K \longrightarrow \mathcal{D}_K$$

which commutes with $\otimes$–products and internal Homs. For any integer there are natural isomorphisms

$$\mathbb{D}^+(V(n)) \cong \mathbb{D}^+(V)(n) \; .$$

For all V we have

$$\mathrm{rk}_{\mathbb{L}^+} \mathbb{D}^+(V) = \dim V \; .$$

$\mathbb{D}^+$ induces an equivalence of categories between $\mathcal{F}il_K$ and $\mathcal{D}_K^{ad}$, the full subcategory of $\mathcal{D}_K$ generated by objects which are isomorphic to finite direct sums of $\mathbb{L}^+(n)$ for n in $\mathbb{Z}$.

Remarks: 1) The subcategory $\mathcal{D}_K^{ad}$ of $\mathcal{D}_K$ is closed under twists by integers.
2) Using the real structure $\mathbb{R}[z, z^{-1}] \subset \mathbb{L}$ on $\mathbb{L}$ the functor $\mathbb{D}^+$ maps real structures on objects of $\mathcal{F}il_K$ to real structures on objects of $\mathcal{D}_K$.

As in [De1] we set $\Gamma_\mathbb{R}(s) = 2^{-1/2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$ and $\Gamma_\mathbb{C}(s) = (2\pi)^{-s}\Gamma(s)$.

(5.2) Proposition: For V in $\mathcal{F}il_K$ set $d_\nu = \dim \mathrm{Gr}^\nu V$. Then we have for all complex s:

$$\dim_\infty\left(\frac{1}{2\pi}(s-\Theta_K)|\mathbb{D}^+(V)\right) = \left(\frac{1}{2}-\frac{s}{e_K}\right)\dim V + \frac{1}{e_K}\sum_\nu \nu d_\nu$$

and

$$\det_\infty\left(\frac{1}{2\pi}(s-\Theta_K)|\mathbb{D}^+(V)\right) = \prod_{\nu\in\mathbb{Z}} \Gamma_K(s-\nu)^{-d_\nu}\ .$$

Proof: Since any object in $\mathcal{F}il_K$ is isomorphic to a direct sum of $\mathbb{C}(n)$'s and since both sides of the equations are additive resp. multiplicative we are reduced to $\mathbb{C}(n)$. Since an n–twist changes s to $s+n$ in all expressions we are reduced to $\mathbb{C}(0)$. Since $\mathbb{D}^+(\mathbb{C}(0)) = \mathbb{L}^+$ and since Θ_K acts on $\mathbb{L}^+$ with eigenvalues $-e_K\nu$ for $\nu = 0, 1, \ldots$ of multiplicity one we have to consider the Dirichlet series

$$\sum_{\nu=0}^{\infty} \frac{1}{\left[\frac{1}{2\pi}(s+e_K\nu)\right]^u} = \zeta_{\frac{e_K}{2\pi}}\left(u, \frac{s}{e_K}\right) \quad , \ \mathrm{Re}\, u > 1$$

in the notation of the proof of (2.7). Now the assertion follows from (2.7.1).

Remark: One may wonder about the factor $(2\pi)^{-1}$ in the above formulas. For the first it is irrelevant but for the second it seems to be the best choice. Namely let δ be in $\mathbb{C}^*$ and set:

$$\Gamma_{\mathbb{C},\delta}(s) = \delta^{s-\frac{1}{2}}\sqrt{2\pi}^{-1}\Gamma(s) \quad \text{and} \quad \Gamma_{\mathbb{R},\delta}(s) = (2\delta)^{\frac{s-1}{2}}\sqrt{2\pi}^{-1}\Gamma\left(\frac{s}{2}\right)\ .$$

Then we have for all complex s:

$$\dim_\infty(\delta(s-\Theta_K)|\mathbb{D}^+(V)) = \left(\frac{1}{2}-\frac{s}{e_K}\right)\dim V + \frac{1}{e_K}\sum_\nu \nu d_\nu$$

and in case δ is not a negative real number:

$$\det_\infty(\delta(s-\Theta_K)|\mathbb{D}^+(V)) = \prod_{\nu\in\mathbb{Z}} \Gamma_{K,\delta}(s-\nu)^{-d_\nu}\ .$$

This follows as above from (2.7.1) and (2.7.2). For $\delta < 0$ the formula for $\det_\infty$ depends on $\mathrm{Im}\, s$. Using the factor $\Gamma_{\mathbb{R},\delta}(s)$ to complete the Riemann zeta function at infinity introduces the ε–factor $(2\pi\delta)^{s-\frac{1}{2}}$ in its functional equation.

6. The archimedian local factors

In the first part of this section we express the archimedian local L–factors of a motive as regularized characteristic power series.

Let $\mathcal{MH}_{\mathbb{C}}$ (resp. $\mathcal{MH}_{\mathbb{R}}$) denote the category of mixed Hodge structures with coefficients in $\mathbb{R}$ (equipped with the action of an infinite Frobenius F_∞ which respects the weight filtration and maps F^i to $\overline{F}^i$) [D2], [Be] §7. Fix a finite extension $E/\mathbb{Q}$ as a field of multiplication. We refer to [F–PR], [De3] or the proof of (6.3) below for the definition of the $E \otimes \mathbb{C}$–valued L–factor $L(H,s)$ of a mixed Hodge structure H in $\mathcal{MH}_K(E)$.

Let us define an additive functor

$$\mathcal{V} : \mathcal{MH}_K \longrightarrow \mathcal{F}il_K \tag{6.1}$$

by

$$\mathcal{V}((H, W_\bullet H, F^\bullet H_{\mathbb{C}})) = \left(H_{\mathbb{C}}, \mathrm{Fil}^i H_{\mathbb{C}} = F^i H_{\mathbb{C}} \cap \overline{F^i H_{\mathbb{C}}}\right)$$

in case $K = \mathbb{C}$ and by

$$\mathcal{V}((H, W_\bullet H, F^\bullet H_{\mathbb{C}}, F_\infty)) = \left(H_{\mathbb{C}}, \mathrm{Fil}^i H_{\mathbb{C}} = \left(F^i H_{\mathbb{C}} \cap \overline{F^i H_{\mathbb{C}}}\right)^{(-1)^i} \oplus \left(F^{i+1} H_{\mathbb{C}} \cap \overline{F^{i+1} H_{\mathbb{C}}}\right)^{(-1)^{i+1}}, F_\infty\right)$$

in case $K = \mathbb{R}$. Here the exponent ± 1 denotes the ± 1–eigenspace of F_∞.

Note that $\mathcal{V}(H)$ carries the real structure

$$\mathcal{V}^{\mathbb{R}}(H) = (H, \mathrm{Fil}^i H = F^i H_{\mathbb{C}} \cap H) \quad \text{in case } K = \mathbb{C}$$

and

$$\mathcal{V}^{\mathbb{R}}(H) = (H, \mathrm{Fil}^i H = (F^i H_{\mathbb{C}} \cap H)^{(-1)^i} \oplus (F^{i+1} H_{\mathbb{C}} \cap H)^{(-1)^{i+1}}, F_\infty) \quad \text{in case } K = \mathbb{R}\,.$$

The functor $\mathcal{V}$ commutes with twists and sends $\mathbb{R}(n)$ in $\mathcal{MH}_K$ to $\mathbb{C}(n)$ in $\mathcal{F}il_K$. For $K = \mathbb{C}$ it commutes with $\otimes$–products but not so for $K = \mathbb{R}$. In both cases $\mathcal{V}$ does not commute with duals.

Composing $\mathcal{V}$ with the functor $\mathbb{D}^+ : \mathcal{F}il_K \to \mathcal{D}_K^{ad}$ we obtain a functor

$$\mathbb{D}^+_{\mathcal{H}} : \mathcal{MH}_K(E) \longrightarrow \mathcal{D}_K^{ad}(E). \tag{6.2}$$

Using the second remark after (5.1) we see that any object $\mathbb{D}^+_{\mathcal{H}}(H)$ in the image of $\mathbb{D}^+_{\mathcal{H}}$ carries a canonical $E \otimes \mathbb{R}$–structure, which we denote by $\mathbb{D}^{+,\mathbb{R}}_{\mathcal{H}}(H)$.

Note that there is a natural isomorphism of categories

$$\mathcal{D}_K(E) \cong \prod_{\sigma \in \mathrm{Hom}(E,\mathbb{C})} \mathcal{D}_K \ , \ D \longmapsto (D_\sigma)$$

where $D_\sigma = D \otimes_{E\otimes\mathbb{C},\sigma} \mathbb{C}$.

Remark: The construction of the functor $\mathbb{D}$ in [De1] §3 uses only the Hodge filtration and not the weight filtration of a Hodge structure. Hence it makes sense for mixed Hodge structures as well and it turns out to be equal to $\mathbb{D}^{+,\mathbb{R}}_{\mathcal{H}}$ above (after the substitution $z = T^{-e_K}$).

(6.3) Proposition. For H in $\mathcal{MH}_K(E)$ and all complex s we have:

$$L(H,s) = \det{}_\infty \left(\frac{1}{2\pi}(s - \Theta_K) \,|\, \mathbb{D}^+_{\mathcal{H}}(H)_\sigma \right)^{-1}_{\sigma \in \mathrm{Hom}(E,\mathbb{C})} .$$

Proof: Let e_σ be the idempotent in $E \otimes \mathbb{C} = \mathbb{C}^{\mathrm{Hom}(E,\mathbb{C})}$ corresponding to the embedding $\sigma : E \hookrightarrow \mathbb{C}$. Then

$$\mathbb{D}^+_{\mathcal{H}}(H)_\sigma = e_\sigma \mathbb{D}^+_{\mathcal{H}}(H) = e_\sigma \mathbb{D}^+(\mathcal{V}(H)) = \mathbb{D}^+(e_\sigma \mathcal{V}(H)) .$$

Note that $\mathbb{D}^+$ is $E \otimes \mathbb{C}$–linear and that $\mathcal{F}il_K(E)$ is pseudo–abelian. We view $e_\sigma\mathcal{V}(H)$ as an object of $\mathcal{F}il_K$. Proposition (5.2) implies:

$$\det{}_\infty \left(\frac{1}{2\pi}(s - \Theta_K) \,|\, \mathbb{D}^+_{\mathcal{H}}(H)_\sigma \right)^{-1} = \prod_{\nu \in \mathbb{Z}} \Gamma_K(s-\nu)^{d_{\nu,\sigma}}$$

where $d_{\nu,\sigma} = \dim_{\mathbb{C}} Gr^\nu(e_\sigma \mathcal{V}(H)) = \dim_{\mathbb{C}}(e_\sigma Gr^\nu \mathcal{V}(H))$.

Consider the filtration γ on $H_{\mathbb{C}}$:

$$\gamma^\nu H_{\mathbb{C}} = F^\nu H_{\mathbb{C}} \cap \overline{F}^\nu H_{\mathbb{C}} ,$$

where F^ν is the Hodge filtration.

For $K = \mathbb{C}$ we have $Gr^\nu \mathcal{V}(H) = Gr^\nu_\gamma H_{\mathbb{C}}$ and since by definition

$$L(H,s)_\sigma = \prod_\nu \Gamma_{\mathbb{C}}(s-\nu)^{n_{\nu,\sigma}}$$

where $n_{\nu,\sigma} = \dim_{\mathbb{C}}(e_\sigma Gr^\nu_\gamma H_{\mathbb{C}})$ the claim follows.

For $K = \mathbb{R}$ set

$$n^{\pm}_{\nu,\sigma} = \dim_{\mathbb{C}} e_\sigma (Gr^\nu_\gamma H_{\mathbb{C}})^{\pm} .$$

Then by definition:

$$\begin{aligned} L(H,s)_\sigma &= \prod_\nu \Gamma_{\mathbb{R}}(s+\varepsilon_\nu-\nu)^{n^+_{\nu,\sigma}} \Gamma_{\mathbb{R}}(s+1-\varepsilon_\nu-\nu)^{n^-_{\nu,\sigma}} \\ &= \prod_\nu \Gamma_{\mathbb{R}}(s-\nu)^{d'_{\nu,\sigma}} \end{aligned}$$

where $\varepsilon_\nu \in \{0,1\}$, $\varepsilon_\nu \equiv \nu \bmod 2$ and

$$d'_{\nu,\sigma} = n^{(-1)^\nu}_{\nu+1,\sigma} + n^{(-1)^\nu}_{\nu,\sigma} .$$

Clearly

$$Gr^\nu \mathcal{V}(H) \cong \left(\frac{\gamma^\nu H_{\mathbb{C}}}{\gamma^{\nu+2} H_{\mathbb{C}}} \right)^{(-1)^\nu}$$

and thus there is an $E \otimes \mathbb{C}$–equivariant exact sequence

$$0 \longrightarrow (Gr^{\nu+1}_\gamma H_{\mathbb{C}})^{(-1)^\nu} \longrightarrow Gr^\nu \mathcal{V}(H) \longrightarrow (Gr^\nu_\gamma H_{\mathbb{C}})^{(-1)^\nu} \longrightarrow 0 .$$

This implies that

$$d'_{\nu,\sigma} = \dim e_\sigma Gr^\nu \mathcal{V}(H) = d_{\nu,\sigma}$$

and hence the assertion.

Up to now we have kept $\mathbb{L}^+$ fixed and chosen Θ_K depending on the ground field K. As in section 3 and in [De1] we can also keep the derivation fixed and vary the spaces:

Let $\tau : K \hookrightarrow \mathbb{C}$ be any embedding and denote by $\mathfrak{p} = \{\tau, \overline{\tau}\}$ the unique place of K. Consider the subring:

$$\mathbb{L}^+_{\mathfrak{p}} = \mathbb{C}[\exp(-e_K \xi)] \subset \mathcal{O}(\mathbb{C})$$

equipped with the derivation $\Theta = \frac{d}{d\xi}$. The change of variables $z = \exp(-e_K\xi)$ identifies the pair $(\mathbb{L}^+, \Theta_K)$ with $(\mathbb{L}^+_{\mathfrak{p}}, \Theta)$. We write $\mathcal{D}_{\mathfrak{p}}$ for the category with twists $\mathcal{D}_K$ of section 5 if we make the identification $(\mathbb{L}^+, \Theta_K) = (\mathbb{L}^+_{\mathfrak{p}}, \Theta)$ in its construction. We can view $\mathbb{D}^+_{\mathcal{H}}$ as a functor

$$\mathbb{D}^+_{\mathcal{H}} : \mathcal{MH}_K(E) \longrightarrow \mathcal{D}^{\mathrm{ad}}_{\mathfrak{p}}(E) .$$

For $K = \mathbb{C}$ or $\mathbb{R}$ the real Betti realization

$$M_B = H^\bullet_{sing}(M \otimes_K \overline{K}, \mathbb{R})$$

of a motive M over K induces a functor

$$\mathcal{MM}_K(E) \longrightarrow \mathcal{MH}_K(E)\ ,\ M \longmapsto M_B\ .$$

Its composition with $\mathbb{D}^+_{\mathcal{H}}$ is denoted by

$$\mathcal{F}_{\mathfrak{p}} : \mathcal{MM}_K(E) \longrightarrow \mathcal{D}^{\mathrm{ad}}_{\mathfrak{p}}(E)\ .$$

It will turn out to be the archimedian analogue of (3.1.1). Note that for any M in $\mathcal{MM}_K(E)$ there is a canonical $E \otimes \mathbb{R}$–structure $\mathcal{F}^{\mathbb{R}}_{\mathfrak{p}}(M)$ on $\mathcal{F}_{\mathfrak{p}}(M)$. For $E = \mathbb{Q}$ the archimedian cohomology $H^\bullet_{ar}(M)$ of the motive M introduced in [De1] is just $\mathcal{F}^{\mathbb{R}}_{\mathfrak{p}}(M)$.

The functor $\mathcal{F}_{\mathfrak{p}}$ is naturally $\mathbb{Z}$–graded via

$$\mathcal{F}^\bullet_{\mathfrak{p}}(M) = \mathcal{F}_{\mathfrak{p}}(H^\bullet(M))$$

and up to natural isomorphisms it commutes with twists. We view $\mathcal{F}_{\mathfrak{p}}(M)$ as a Lie algebra representation of $\mathfrak{t}$ by sending 1 to Θ.

(6.4) Since $\mathcal{F}_{\mathfrak{p}}$ does not commute with duals and for $K = \mathbb{R}$ not with $\otimes$–products as well we cannot view $\mathcal{F}_{\mathfrak{p}}$ as a *geometric* cohomology theory on motives. This is in accordance with the philosophy of Arakelov theory [Ma] that the “reduction” of the infinite fibres of an arithmetic variety should be viewed as totally degenerate. So $\mathcal{F}_{\mathfrak{p}}(M)$ would be something like the “fixed module under inertia” of the “true cohomology” of M over K. This point of view is compatible with the fact that $\mathcal{F}_{\mathfrak{p}}$ is left exact but not exact and with a later argument about weights (7.14).

As a trivial consequence of (6.3) we get:

(6.5) Corollary: For M in $\mathcal{MM}_K(E)$ and all complex s we have:

$$L_K(M,s) = \det{}_\infty\left(\frac{1}{2\pi}(s-\Theta)\,|\,\mathcal{F}_{\mathfrak{p}}(M)_\sigma\right)^{-1}_{\sigma\in\mathrm{Hom}(E,\mathbb{C})}\ .$$

We now discuss a canonical pairing between the kernel of Θ_K on $\mathbb{D}^{+,\mathbb{R}}_{\mathcal{H}}$ and certain Ext–groups in the category of real mixed Hodge structures over K. For H in $\mathcal{MH}_K$ we have if

$K = \mathbb{C}$

$$\begin{aligned}
\mathbb{D}_{\mathcal{H}}^{+,\mathbb{R}}(H)^{\Theta=0} &= \mathrm{Fil}^0(\mathcal{V}^{\mathbb{R}}(H) \otimes_{\mathbb{R}} \mathbb{R}[z, z^{-1}])^{\Theta=0} \\
&= \left(\sum_{\nu+\mu \geq 0} \mathrm{Fil}^{\nu}\mathcal{V}^{\mathbb{R}}(H) \otimes_{\mathbb{R}} z^{\mu}\mathbb{R}[z] \right)^{\Theta=0} \\
&= \mathrm{Fil}^0\mathcal{V}^{\mathbb{R}}(H) = F^0 H_{\mathbb{C}} \cap H
\end{aligned}$$

and if $K = \mathbb{R}$

$$\begin{aligned}
\mathbb{D}_{\mathcal{H}}^{+,\mathbb{R}}(H)^{\Theta=0} &= (\mathrm{Fil}^0\mathcal{V}^{\mathbb{R}}(H))^{F_\infty=\mathrm{id}} = (F^0 H_{\mathbb{C}} \cap H)^{F_\infty=\mathrm{id}} \\
&= (F^0 H_{\mathbb{C}} \cap H)^{\overline{F}_\infty=\mathrm{id}} .
\end{aligned}$$

For any Hodge structure H define $\pi_n : H_{\mathbb{C}} \to H_{\mathbb{C}}$ by $\pi_n(h) = \frac{1}{2}(h + (-1)^n \overline{h})$ where $- = \mathrm{id}_H \otimes c$. Thus $\pi_n(H_{\mathbb{C}}) = (2\pi i)^n H$. The canonical pairing:

$$H_{\mathbb{C}} \times H^*(1)_{\mathbb{C}} \xrightarrow{\langle,\rangle} \mathbb{R}(1)_{\mathbb{C}} \xrightarrow{\pi_1} \mathbb{R}$$

induces an $\mathbb{R}$–linear pairing:

(6.6)
$$(F^0 H_{\mathbb{C}} \cap H) \times \frac{H^*(1)_{\mathbb{C}}}{H^*(1) + F^0 H^*(1)_{\mathbb{C}}} \longrightarrow \mathbb{R}$$

since $\langle H, H^*(1) \rangle \subset \mathbb{R}(1) \subset \mathrm{Ker}\, \pi_1$ and $\langle F^0 H_{\mathbb{C}}, F^0 H^*(1)_{\mathbb{C}} \rangle \subset F^0 \mathbb{R}(1)_{\mathbb{C}} = 0$. There are canonical isomorphisms:

$$\begin{aligned}
\frac{H^*(1)_{\mathbb{C}}}{H^*(1) + F^0 H^*(1)_{\mathbb{C}}} &= \frac{H^*(1)_{\mathbb{C}}/H^*(1)}{F^0 H^*(1)_{\mathbb{C}}/(F^0 H^*(1)_{\mathbb{C}} \cap H^*(1))} \overset{\pi_1}{\cong} \frac{H^*}{\pi_1(F^0 H^*(1)_{\mathbb{C}})} \\
&= \frac{H^*}{\pi_0(F^1 H^*_{\mathbb{C}})}
\end{aligned}$$

and (6.6) translates into the pairing

(6.7)
$$(F^0 H_{\mathbb{C}} \cap H) \times \frac{H^*}{\pi_0(F^1 H^*_{\mathbb{C}})} \longrightarrow \mathbb{R}$$

induced by the canonical pairing $H \times H^* \to \mathbb{R}$.

We claim that (6.7) and hence (6.6) is non–degenerate:

$$\left(\frac{H^*}{\pi_0(F^1 H^*_{\mathbb{C}})} \right)^* = \mathrm{Ker}\,(H \longrightarrow (\pi_0(F^1 H^*_{\mathbb{C}}))^*)$$

consists of all h in H such that $\mathrm{Re}(\psi(h)) = 0$ for all $\mathbb{C}$–linear maps $\psi : H_{\mathbb{C}} \to \mathbb{C}$ with $\psi(F^0(H_{\mathbb{C}})) = 0$. Replacing ψ by $i\psi$ we find that in fact $\psi(h) = 0$ for all such ψ and hence that h is in $F^0 H_{\mathbb{C}}$. Thus we have

$$\left(\frac{H^*}{\pi_0(F^1 H^*_{\mathbb{C}})}\right)^* = F^0 H_{\mathbb{C}} \cap H$$

as claimed.

In case $K = \mathbb{R}$ the de Rham conjugation $\overline{F}_\infty$ is selfadjoint with respect to the pairing in (6.6). Hence the pairing of fixed modules

$$(F^0 H_{\mathbb{C}} \cap H)^{\overline{F}_\infty} \times \left(\frac{H^*(1)_{\mathbb{C}}}{H^*(1) + F^0 H^*(1)_{\mathbb{C}}}\right)^{\overline{F}_\infty} \longrightarrow \mathbb{R}$$

is non–degenerate as well.

Now recall [Be] (1.4), (1.6) that there are canonical isomorphisms

$$\begin{aligned} \mathrm{Ext}^1_{\mathcal{MH}_{\mathbb{C}}}(\mathbb{R}(0), H) &\cong \frac{W_0 H_{\mathbb{C}}}{W_0 H + F^0 W_0 H_{\mathbb{C}}} \qquad \text{and} \\ \mathrm{Ext}^1_{\mathcal{MH}_{\mathbb{R}}}(\mathbb{R}(0), H) &\cong \left(\frac{W_0 H_{\mathbb{C}}}{W_0 H + F^0 W_0 H_{\mathbb{C}}}\right)^{\overline{F}_\infty} . \end{aligned}$$

Thus we have proved:

(6.8) Proposition: For any H in $\mathcal{MH}_K$ and M in $\mathcal{MM}_K$ there are canonical pairings of finite dimensional $\mathbb{R}$–vector spaces:

$$\begin{aligned} \mathbb{D}^{+,\mathbb{R}}_{\mathcal{H}}(H)^{\Theta=0} &\times \mathrm{Ext}^1_{\mathcal{MH}_K}(\mathbb{R}(0), H^*(1)) \longrightarrow \mathbb{R} \\ \mathcal{F}^{\mathbb{R}}_{\mathfrak{p}}(M)^{\Theta=0} &\times \mathrm{Ext}^1_{\mathcal{MH}_K}(\mathbb{R}(0), M^*_B(1)) \longrightarrow \mathbb{R} \end{aligned}$$

which are non–degenerate if the weights of H resp. M are ≥ -2.

Now let us use the proposition to clarify the relation between archimedian and Deligne cohomology. Let X/K be a smooth projective variety of dimension d. Then we have canonical isomorphisms

$$\begin{aligned} H^{w+1}_{\mathcal{D}}(X, \mathbb{R}(n)) &\cong \mathrm{Ext}^1_{\mathcal{MH}_K}(\mathbb{R}(0), H^w_B(X)(n)) \quad \text{if } w+1 < 2n \\ &\cong (\mathcal{F}^{\mathbb{R}}_{\mathfrak{p}}(H^w(X)^*(1-n))^{\Theta=0})^* . \end{aligned}$$

The induced isomorphism for $w+1<2n$

$$(6.9) \qquad \alpha : H^{w+1}_{\mathcal{D}}(X, \mathbb{R}(n)) \longrightarrow (\mathcal{F}^{\mathbb{R}}_{\mathfrak{p}}(H^{2d-w}(X)(d+1-n))^{\Theta=0})^*$$

will play a role in section 7. If we compose with the inverse of a strong Lefschetz isomorphism $H^w(X) \xrightarrow{\sim} H^{2d-w}(X)(d-w)$ we get the first part of

(6.10) Proposition: For $m = w+1-n \le \frac{w}{2}$ there is a natural perfect pairing

$$H^{w+1}_{\mathcal{D}}(X, \mathbb{R}(n)) \times \mathcal{F}^{\mathbb{R}}_{\mathfrak{p}}(H^w(X))^{\Theta=m} \longrightarrow \mathbb{R}\,.$$

For $m > \frac{w}{2}$ the groups $\mathcal{F}^{\mathbb{R}}_{\mathfrak{p}}(H^w(X))^{\Theta=m}$ vanish. In particular the weights of Θ on $\mathcal{F}^{\mathbb{R}}_{\mathfrak{p}}(H^w(X))$ are $\le w$.

By (6.5) it is clear that

$$(6.11) \qquad \mathrm{ord}_{s=m} L_K(H^w(X), s) = -\dim_{\mathbb{R}} \mathcal{F}^{\mathbb{R}}_{\mathfrak{p}}(H^w(X))^{\Theta=m} \quad \text{for } m \text{ in } \mathbb{Z}.$$

Since $L_K(H^w(X), s)$ has no poles for $s > \frac{w}{2}$ the second assertion of the proposition follows.

Note that the duality in (6.10) gives a satisfactory algebraic explanation for Beilinson's observation that

$$(6.12) \qquad \mathrm{ord}_{s=m} L_K(H^w(X), s) = -\dim_{\mathbb{R}} H^{w+1}_{\mathcal{D}}(X, \mathbb{R}(n)) \quad \text{if } m \le \tfrac{w}{2}.$$

The present explanation of (6.12) is to be preferred to the one in [De1] §5.

7. Arithmetic cohomology?

In this section we extensively discuss aspects of the still speculative site $\mathcal{S}$ of [De2] §3 (called $\mathcal{T}$ in loc. cit.). The discussion is meant as an approximation, so on occasion we will deliberately be somewhat vague. The motives attached to algebraic Hecke characters will serve us to test consequences of our considerations. In some cases we are suggested new formulas in analytic number theory which can be proved by classical methods (7.16), (7.18), (7.20).

$\mathcal{S}$ should be a site ringed by a sheaf of $\mathbb{C}$–algebras $\mathcal{C}$. The objects in the underlying category of $\mathcal{S}$ should be equipped with functorial actions of the group $T = (\mathbb{R}, +)$. Schemes and

Arakelov varieties should give rise to objects of cat $\mathcal{S}$ which we denote by the same symbol. There should be a morphism of ringed sites from varieties over $\mathbb{C}$ with the analytic topology and ringed by $\mathbb{C}$ into $\mathcal{S}$. For any embedding $\sigma : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ a $\mathbb{Q}_\ell$–sheaf on a variety over a field of characteristic zero should give rise to a $\mathcal{C}$–sheaf on the corresponding object of cat $\mathcal{S}$.

Because of the T–actions the cohomology groups $H^w(X, \mathcal{C})$ of any object in cat $\mathcal{S}$ would be $\mathbb{C}$–vector spaces with an action of T. We expect them to decompose into the direct sum of finite dimensional T–spaces. Thus they would carry an action by the Lie–algebra $\mathfrak{t}$ of T and hence an action by the endomorphism Θ corresponding to $1 \in \mathfrak{t} = \mathbb{R}$.

If $X \xrightarrow{\pi} Y$ is a morphism, cup product would turn $H^w(X, \mathcal{C})$ into an $H^0(Y, \mathcal{C})$–module. For a non–archimedian local field K of characteristic zero we expect $H^0(\mathrm{Spec}\, K, \mathcal{C}) \cong \mathbb{L}_\mathfrak{p}$ in $\mathcal{D}_\mathfrak{p}$. If X is a smooth projective variety over K with good reduction, we now expect:

$$H^w(X, \mathcal{C}) \cong H^w(X/\mathbb{L}_\mathfrak{p}) \quad \text{in } \mathcal{D}_\mathfrak{p} \, .$$

There is a similarity to the situation for crystalline cohomology: coefficients vary with the ground field and the action of T which corresponds to $\mathbb{Z} \cong \langle Fr \rangle$ is σ–linear with respect to the coefficient module structure on cohomology. In our situation the action of T on $\mathbb{L}_\mathfrak{p}$ is given by $\sigma : T \to \mathrm{Aut}_{\mathbb{C}} \mathbb{L}_\mathfrak{p}$, $(\sigma(t)\ell)(\xi) = \ell(\xi + t)$.

A very optimistic suggestion to explain the T–action would be the following: There should be a "ground point" P in $\mathcal{S}$ (which is not a scheme) and an extension $\hat{P} \to P$ with $\mathrm{Aut}_P(\hat{P}) \cong T$. What was written $H^w(X, \mathcal{C})$ above should in fact be $H^w(X \times_P \hat{P}, \mathcal{C})$ with the T–action coming from transport of structure. This guess may be too naive: For reasons which become clear later when we discuss zeta–functions we would have $H^0(\hat{P}, \mathcal{C}) \cong \mathbb{C}$ with trivial T–action. If $\mathrm{Spec}\, K \to P$ were geometrically connected (with respect to $- \times_P \hat{P}$) I would expect

$$H^0(\mathrm{Spec}\, K \times_P \hat{P}, \mathcal{C}) \cong H^0(\hat{P}, \mathcal{C}) \cong \mathbb{C}$$

a contradiction. Hence $\mathrm{Spec}\, K \times_P \hat{P}$ should not be connected. But then $H^0(\mathrm{Spec}\, K \times_P \hat{P}, \mathcal{C}) \cong \mathbb{L}_\mathfrak{p}$ would not be an integral domain which is also a contradiction. So the crystalline picture may be more appropriate than this analogy with the étale site over a finite field.

For any local field K, we expect to have a model $\mathcal{Y}_K$ of $\mathrm{Spec}\, K$ in $\mathcal{S}$ with closed point $\mathfrak{p}$. For non–archimedian K , $\mathcal{Y}_K$ would be $\operatorname{Spec}\mathcal{O}_K$. For any motive M in $\mathcal{MM}_K(E)$ we expect to have a canonical T–sheaf $\mathcal{F}(M)$ with E–action on the $\mathcal{S}$–site of $\mathcal{Y}_K$ such that its stalk at $\mathfrak{p}$ is given by

$$\mathcal{F}(M)_{\mathfrak{p}} = H^0(\mathfrak{p}, \mathcal{F}(M)) = \mathcal{F}_{\mathfrak{p}}(M)$$

in $\mathcal{D}_{\mathfrak{p}}(E)$ with the objects $\mathcal{F}_{\mathfrak{p}}(M)$ of §§3, 6.

If k is a number field, let $\mathcal{Y}$ be the object in $\mathcal{S}$ corresponding to the Arakelov compactification $\operatorname{Spec}\mathcal{O}_k \cup \{\mathfrak{p}|\infty\}$ of $\operatorname{Spec}\mathcal{O}_k$. As in the local case we expect for any M in $\mathcal{MM}_k(E)$ a canonical T–sheaf $\mathcal{F}(M)$ with endomorphisms in E on the $\mathcal{S}$-site of $\mathcal{Y}$ such that

$$\mathcal{F}(M)_{\mathfrak{p}} \cong \mathcal{F}_{\mathfrak{p}}(M) := \mathcal{F}_{\mathfrak{p}}(M \otimes_k k_{\mathfrak{p}}) \quad \text{in } \mathcal{D}_{\mathfrak{p}}(E) \tag{7.1}$$

for all places $\mathfrak{p}$ of k. The formation of $\mathcal{F}$ should commute with twists and duals.

If for example $M = H^w(X)$ where $\pi : X \to \operatorname{Spec} k$ is a smooth and projective variety over k we expect

$$\mathcal{F}(M) = j_* R^w \pi_* \mathcal{C}_X \tag{7.1.1}$$

where $j : \operatorname{Spec} k \hookrightarrow \mathcal{Y}$ is the inclusion. For the motive $\mathbb{Q}(0) = H^0(\operatorname{Spec} k)$ in $\mathcal{M}_k$ we expect

$$\mathcal{F}(\mathbb{Q}(0)) = j_* \mathcal{C}_{\operatorname{Spec} k} = \mathcal{C}_{\mathcal{Y}}. \tag{7.1.2}$$

For suitable analytic functions Φ a functional calculus in the algebra of correspondences should allow one to construct $\Phi(\Theta)$ as a correspondence on $\mathcal{Y}$. Let S be a finite set of places of k and set $\mathcal{Y}_S = \mathcal{Y} \setminus S$. If Φ is such that $\Phi(\Theta)$ is of trace class on all stalks $\mathcal{F}(M)_{\mathfrak{p}}$ and on the cohomologies $H^i_c(\mathcal{Y}_S, \mathcal{F}(M))$ of $\mathcal{Y}_S$ with compact support, we are lead to expect the following Lefschetz trace formula:

$$\sum_{\mathfrak{p} \in |\mathcal{Y}_S|} \operatorname{Tr}_E(\Phi(\Theta) | \mathcal{F}(M)_{\mathfrak{p}}) = \sum_{i=0}^{2} (-1)^i \operatorname{Tr}_E(\Phi(\Theta) | H^i_c(\mathcal{Y}_S, \mathcal{F}(M))) \,. \tag{7.2}$$

Here $|\mathcal{Y}_S|$ are the finite and infinite places of k that are not contained in S. For an endomorphism φ of an $E \otimes \mathbb{C}$–module V we set:

$$\operatorname{Tr}_E(\varphi | V) = (\operatorname{Tr}(\varphi | V_\sigma))_{\sigma \in \operatorname{Hom}(E, \mathbb{C})} \in E \otimes \mathbb{C}$$

if the traces of φ on all $V_\sigma = V \otimes_{E \otimes \mathbb{C}, \sigma} \mathbb{C}$ exist.

Now let us fix z, s with real parts sufficiently large and consider $\Phi(\tau) = (s - \tau)^{-z}$. Let

$\zeta_{\mathfrak{p},\sigma}(z)$ resp. $\zeta_{i,\sigma}(z)$ denote the zeta functions of the operator $\frac{1}{2\pi}(s-\Theta)$ on $\mathcal{F}_{\mathfrak{p}}(M)_\sigma$ resp. on $H^i_c(\mathcal{Y}_S,\mathcal{F}(M))_\sigma$. Then (7.2) combined with (7.1) implies that:

$$\sum_{\mathfrak{p}\in|\mathcal{Y}_S|}\zeta_{\mathfrak{p},\sigma}(z)=\sum_{i=0}^{2}(-1)^i\zeta_{i,\sigma}(z)\,. \tag{7.3}$$

If the convergence on the left is locally uniform in $\operatorname{Re}z>-\varepsilon$, $\varepsilon>0$ we obtain:

$$\sum_{\mathfrak{p}\in|\mathcal{Y}_S|}\zeta'_{\mathfrak{p},\sigma}(0)=\sum_{i=0}^{2}(-1)^i\zeta'_{i,\sigma}(0)$$

and hence by definition of regularized determinants:

$$\prod_{\mathfrak{p}\in|\mathcal{Y}_S|}\det\nolimits_\infty\left(\frac{1}{2\pi}(s-\Theta)|\mathcal{F}_{\mathfrak{p}}(M)_\sigma\right)^{-1}=\prod_{i=0}^{2}\det\nolimits_\infty\left(\frac{1}{2\pi}(s-\Theta)|H^i_c(\mathcal{Y}_S,\mathcal{F}(M))_\sigma\right)^{(-1)^{i+1}}\,.$$

According to (3.2) and (6.5) the left hand side equals

$$\hat{L}_S(M,s)_\sigma=\prod_{\mathfrak{p}\in|\mathcal{Y}_S|}L_{\mathfrak{p}}(M,s)_\sigma$$

where $L_{\mathfrak{p}}(M,s)=L_{k_{\mathfrak{p}}}(M\otimes_k k_{\mathfrak{p}},s)$ in earlier notation and where we have put the hat on L to indicate the possible presence of archimedian factors. Thus we finally get:

$$\hat{L}_S(M,s)=\prod_{i=0}^{2}\det\nolimits_\infty\left(\frac{1}{2\pi}(s-\Theta)|H^i_c(\mathcal{Y}_S,\mathcal{F}(M))\right)^{(-1)^{i+1}} \tag{7.4}$$

for $\operatorname{Re}s$ large enough.

I expect that the expressions

$$\det\nolimits_\infty\left(\frac{1}{2\pi}(s-\Theta)|H^i_c(\mathcal{Y}_S,\mathcal{F}(M))\right)$$

extend to entire functions. If $\det_\infty$ were defined via Weierstra products this would be clear. For regularized determinants I do not know whether this property is automatic without further regularity conditions as in [C–V].

The zeros of $\hat{L}_S(M,s)$ would be eigenvalues of Θ on $H^1_c(\mathcal{Y}_S,\mathcal{F}(M))$ and

$$\det\nolimits_\infty\left(\frac{1}{2\pi}(s-\Theta)|H^1_c(\mathcal{Y}_S,\mathcal{F}(M))\right)$$

should have order one as an entire function.

(7.5) If M is a pure motive of weight w in $\mathcal{M}_k(E)$ then by analogy with [D4] Th. (3.2.3) and taking (7.1.1) into account we expect that $H^i(\mathcal{Y},\mathcal{F}(M))$ is pure of weight $w+i$ i.e. that the eigenvalues of Θ have real part $\frac{w+i}{2}$. From the expression

$$\hat{L}(M,s) = \prod_{i=0}^{2} \det{}_\infty \left(\frac{1}{2\pi}(s-\Theta)|H^i(\mathcal{Y},\mathcal{F}(M))\right)^{(-1)^{i+1}} \tag{7.6}$$

we would then get: The poles of $\hat{L}(M,s)$ are exactly the eigenvalues of Θ on the finite dimensional spaces $H^i(\mathcal{Y},\mathcal{F}(M))$ for $i=0$ and 2; they have real part $\frac{w}{2}$ or $\frac{w}{2}+1$. The zeros of $\hat{L}(M,s)$ are exactly the eigenvalues of Θ on the infinite dimensional space $H^1(\mathcal{Y},\mathcal{F}(M))$ and hence should have real part $\frac{w+1}{2}$. That the zeros of $\hat{L}(M,s)$ should lie on the line of symmetry $\operatorname{Re} s = \frac{w+1}{2}$ for the (expected) functional equation of $\hat{L}(M,s)$ is of course a generalization of Riemann's conjecture to pure motives. It is compatible with the Riemann conjecture for Dirichlet L–series which is commonly considered in analytic number theory and with the investigations about non–trival zeros of automorphic L–functions in [Mo].

(7.7) Example: Let χ be an algebraic Hecke character over the number field k with values in the number field E see e.g. [Sch]. For any embedding σ of E into $\mathbb{C}$ let

$$L(\chi^\sigma,s) = \prod_{\mathfrak{p}\not| \mathfrak{f}_\chi} (1-\chi(\mathfrak{p})N\mathfrak{p}^{-s})^{-1} \quad \text{where } \mathfrak{f}_\chi \text{ is the conductor of } \chi$$

be the L–series of $\chi^\sigma = \sigma\chi$. We set

$$\hat{L}(\chi^\sigma,s) = L(\chi^\sigma,s)\prod_{\mathfrak{p}|\infty} L_{\mathfrak{p}}(\chi^\sigma,s)$$

where the local factors at infinity $L_{\mathfrak{p}}(\chi^\sigma,s)$ are defined as follows: Let χ^σ have infinity type $\sum_\tau n_\tau(\sigma)\cdot\tau \in \mathbb{Z}[\operatorname{Hom}(k,\mathbb{C})]$ i.e.

$$\chi^\sigma((\alpha)) = \prod_\tau (\alpha^\tau)^{n_\tau(\sigma)} \quad \text{for totally positive } \alpha \in k^* \ , \ \alpha \equiv 1 \bmod \mathfrak{f}_\chi \ .$$

Let $w = n_\tau(\sigma) + n_{\overline{\tau}}(\sigma)$ denote the weight of χ.

If $\mathfrak{p}$ is a real place of k (whose existence implies that all $n_\tau(\sigma) = \frac{w}{2}$), put

$$L_{\mathfrak{p}}(\chi^\sigma,s) = \Gamma_{\mathbb{R}}\left(s + \varepsilon_{\mathfrak{p}}^\sigma - \frac{w}{2}\right)$$

where $\varepsilon_{\mathfrak{p}} \in \{0,1\}$ is such that the $\mathfrak{p}$–component $\chi_{\mathfrak{p}}^\sigma : k_{\mathfrak{p}}^* \to \mathbb{C}^*$ of the idèle class character attached to χ^σ satisfies $\chi_{\mathfrak{p}}^\sigma(-1) = (-1)^{\varepsilon_{\mathfrak{p}}^\sigma + \frac{w}{2}}$.

If $\mathfrak{p}$ is a complex place corresponding to the pair $\tau,\overline{\tau} : k \hookrightarrow \mathbb{C}$ we put

$$L_{\mathfrak{p}}(\chi^\sigma,s) = \Gamma_{\mathbb{C}}(s - \min(n_\tau(\sigma), n_{\overline{\tau}}(\sigma))) \ .$$

Let $M(\chi)$ in $\mathcal{M}_k(E)$ denote the motive of χ in the sense of [D3] §8 and [Sch] Ch. I Th. 4.1. See also [De–Mu] (3.4). Then we have:

$$L(M(\chi),s) = (L(\chi^\sigma,s))_{\sigma\in \mathrm{Hom}(E,\mathbb{C})} \quad \text{and} \quad \hat{L}(M(\chi),s) = (\hat{L}(\chi^\sigma,s))_{\sigma\in \mathrm{Hom}(E,\mathbb{C})} \; .$$

If $\chi \neq N^n_{k/\mathbb{Q}}$ for all $n\in\mathbb{Z}$ then each L–series $\hat{L}(\chi^\sigma,s)$ is an entire function and according to the discussion in (7.5) we are lead to expect that

$$H^i(\mathcal{Y},\mathcal{F}(M(\chi))) = 0 \quad \text{for} \quad i=0,2. \tag{7.8}$$

and hence that

$$\hat{L}(M(\chi),s) = \det{}_\infty\left(\frac{1}{2\pi}(s-\Theta)|H^1(\mathcal{Y},\mathcal{F}(M))\right) \; . \tag{7.8.1}$$

As regards the trival character we have

$$L(\mathbb{Q}(0),s) = \zeta_k(s) \quad \text{and} \quad \hat{L}(\mathbb{Q}(0),s) = \zeta_k(s)\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2} \; .$$

Thus $\hat{L}(\mathbb{Q}(0),s)$ has first order poles at $s=0,1$ and is holomorphic in $\mathbb{C}\setminus\{0,1\}$. Because of (7.1.2) and (7.5) we therefore expect

$$\begin{array}{rcccl} H^0(\mathcal{Y},\mathcal{C}) & = & H^0(\mathcal{Y},\mathcal{F}(\mathbb{Q}(0))) & = & \mathbb{C}(0) \\ H^2(\mathcal{Y},\mathcal{C}) & = & H^2(\mathcal{Y},\mathcal{F}(\mathbb{Q}(0))) & \overset{\mathrm{Tr}}{\cong} & \mathbb{C}(-1) \end{array} \tag{7.9}$$

as $\mathbb{C}[\Theta]$–modules, where $\mathbb{C}(\alpha)$ is $\mathbb{C}$ with Θ acting by $-\alpha$ id i.e. $\Theta_{\mathbb{C}(\alpha)} = \Theta_{\mathbb{C}(0)} - \alpha$ id.

For the ξ–function of k:

$$\xi_k(s) = \frac{s}{2\pi}\;\frac{s-1}{2\pi}\zeta_k(s)\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}$$

formula (7.6) now implies:

$$\xi_k(s) = \det{}_\infty\left(\frac{1}{2\pi}(s-\Theta)|H^1(\mathcal{Y},\mathcal{C})\right). \tag{7.10}$$

The case where $\chi = N^n_{k/\mathbb{Q}}$ follows by twisting with $-n$.

(7.11) Assuming the existence of suitable Hodge $*$–operators the conjectures on weights in (7.5) would ensue by adapting Serre's argument in [Se] as follows: Assume $E=\mathbb{Q}$ for simplicity. For a pure motive M in $\mathcal{M}_k$ of weight w let M^* denote the dual motive. Cup product and the trace map (7.9) would give a T–equivariant pairing:

$$H^i(\mathcal{Y},\mathcal{F}(M)) \times H^{2-i}(\mathcal{Y},\mathcal{F}(M)^*(1-i-w))) \xrightarrow{\cup} H^2(\mathcal{Y},\mathcal{F}(\mathbb{Q}(1-i-w)) \xrightarrow[\sim]{\mathrm{Tr}} \mathbb{C}(-i-w) \; .$$

Hence Θ would behave as a derivation with respect to $\cup$. For f_1 in $H^i(\mathcal{Y},\mathcal{F}(M))$ and f_2 in $H^{2-i}(\mathcal{Y},\mathcal{F}(M)^*(1-i-w)))$ we would get:

$$(w+i)f_1 \cup f_2 = \Theta(f_1 \cup f_2) = \Theta f_1 \cup f_2 + f_1 \cup \Theta f_2 \,. \tag{7.12}$$

Now assume the existence of a $\mathbb{C}$–antilinear isomorphism

$$* : H^i(\mathcal{Y},\mathcal{F}(M)) \xrightarrow{\sim} H^{2-i}(\mathcal{Y},\mathcal{F}(M)^*(1-i-w))$$

which is T – and hence also $\mathfrak{t}$–equivariant such that the hermitian bilinear form on $H^i(\mathcal{Y},\mathcal{F}(M))$ defined by

$$\langle f, f'\rangle = \mathrm{Tr}(f \cup *f')$$

is positive definite, c.f. [Well] V.2.

Because of $\Theta \circ * = * \circ \Theta$ relation (7.12) implies

$$(w+i)\langle f, f'\rangle = \langle \Theta f, f'\rangle + \langle f, \Theta f'\rangle \tag{7.13}$$

for all f, f' in $H^i(\mathcal{Y},\mathcal{F}(M))$. If $\Theta f = \rho f$ for some $f \neq 0$ setting $f' = f$ implies

$$(w+i)\|f\|^2 = \rho\|f\|^2 + \overline{\rho}\|f\|^2 \quad \text{i.e. } \mathrm{Re}\,\rho = \frac{w+i}{2} \,.$$

Note that (7.13) is equivalent to the relation $\Theta = \frac{w+i}{2} + iS$ where S is symmetric. The completion of $H^i(\mathcal{Y},\mathcal{F}(M))$ with respect to $\langle,\rangle$ would be a Hilbert space with a T–action and an unbounded operator Θ satisfying (7.13) on its domain of definition.

(7.14) We leave the discussion of weights with one more remark: The weight of Θ on $H^0(\mathrm{Spec}\,\mathbb{Q}_p, \mathcal{C}) = \mathbb{L}_p$ is zero for $p < \infty$. For $p = \infty$ the weights of Θ on

$$\mathbb{L}_\infty \cong \mathcal{F}_\infty(H^0(\mathrm{Spec}\,(\mathbb{R})) \stackrel{?}{=} H^0(\infty, \mathcal{C}) \stackrel{?}{=} \text{“}H^0(\mathrm{Spec}\,(\mathbb{R}), \mathcal{C})^{I_\infty}\text{”}$$

are $0, -4, -8, \ldots$. Thus $\mathrm{Spec}\,\mathbb{Q}_p$ should be “smooth, proper” in the geometry of cat $\mathcal{S}$ for $p < \infty$ whereas $\mathrm{Spec}\,\mathbb{R}$ would have totally degenerate reduction ∞. This argument extends to arbitrary motives and it is compatible with Manin's point of view in [Ma].

(7.15) Let us turn to some evidence in favour of the above formalism for the motives $M(\chi)$ attached to algebraic Hecke characters as in (7.7). We first discuss the Lefschetz trace formula (7.2): According to (3.2) and (6.5) the eigenvalues of Θ on $\mathcal{F}_{\mathfrak{p}}(M(\chi))_\sigma$ with their algebraic multiplicities equal the poles with their order of

$$L_{\mathfrak{p}}(M(\chi),s)_\sigma = L_{\mathfrak{p}}(\chi^\sigma,s)$$

i.e. the numbers

- $(\log N\mathfrak{p})^{-1}(\log\chi^\sigma(\mathfrak{p})+2\pi i\nu)$ for $\nu\in\mathbb{Z}$ if $\mathfrak{p}\not|\mathfrak{f}_\chi$ is finite,
- $\frac{w}{2}-\varepsilon_{\mathfrak{p}}^\sigma-2\nu$ for $\nu\geq 0$ if $\mathfrak{p}$ is real,
- $\min(n_\tau(\sigma),n_{\overline{\tau}}(\sigma))-\nu$ for $\nu\geq 0$ if $\mathfrak{p}=\{\tau,\overline{\tau}\}$ is complex,

in the notation of (7.7). Hence $\mathrm{Tr}(\Phi(\Theta)|\mathcal{F}_{\mathfrak{p}}(M(\chi))_\sigma)$ is equal to

- $\sum_{\nu\in\mathbb{Z}}\Phi((\log N\mathfrak{p})^{-1}(\log\chi^\sigma(\mathfrak{p})+2\pi i\nu))$ for finite $\mathfrak{p}\not|\mathfrak{f}_\chi$,
- zero for $\mathfrak{p}|\mathfrak{f}_\chi$,
- $\sum_{\nu=0}^{\infty}\Phi\left(\frac{w}{2}-\varepsilon_{\mathfrak{p}}^\sigma-2\nu\right)$ for real $\mathfrak{p}$,
- $\sum_{\nu=0}^{\infty}\Phi(\min(n_\tau(\sigma),n_{\overline{\tau}}(\sigma))-\nu)$ for complex $\mathfrak{p}=\{\tau,\overline{\tau}\}$

if the sums converge.

According to (7.2), (7.8), (7.8.1) and (7.9), (7.10) we expect that

$$\sum_{i=0}^{2}(-1)^i\mathrm{Tr}(\Phi(\Theta)|H^i(\mathcal{Y},\mathcal{F}(M(\chi)))_\sigma) \quad \text{equals}$$

- $-\sum_\rho\Phi(\rho)$ if $\chi\neq N_{k/\mathbb{Q}}^n$ for all $n\in\mathbb{Z}$,
- $\Phi(0)-\sum_\rho\Phi(\rho)+\Phi(1)$ if $\chi=1$

if the sums over the non–trivial zeros ρ of $L(\chi^\sigma,s)$ converge. The following theorem which is proved in [De4] Cor. (1.8) is therefore compatible with (7.2).

(7.16) Theorem: Fix some $\frac{w}{2}+1<a$ and consider a holomorphic function $\Phi(z)$ in an open subset of $\mathbb{C}$ containing $\mathrm{Re}\,z\leq a$. Assume that there exist constants $c_1>0,c_2>0,\alpha>1$ such that

$$|\Phi(x+iy)|\leq c_1(|y|+c_2)^{-\alpha} \quad \text{for } x\leq a \text{ and } y\in\mathbb{R}\,.$$

Then the above sums for the local traces $\mathrm{Tr}(\Phi(\Theta)|\mathcal{F}_{\mathfrak{p}}(M(\chi))_\sigma)$ are absolutely convergent and we have an identity of absolutely convergent series:

$$\sum_{\mathfrak{p}} \mathrm{Tr}(\Phi(\Theta)|\mathcal{F}_{\mathfrak{p}}(M(\chi))_{\sigma}) = \delta(\Phi(0)+\Phi(1)) - \sum_{\rho} \Phi(\rho)$$

where ρ runs over the non–trivial zeros of $L(\chi^{\sigma}, s)$ and $\delta = 0$ if $\chi \neq N^n_{k/\mathbb{Q}}$ for all $n \in \mathbb{Z}$ and $\delta = 1$ if $\chi = 1$.

A more general result is given in [De4] Thm. (1.7).

Note that the function $\Phi(\tau) = (s-\tau)^{-z}$ is included for $\mathrm{Re}\, z > 1$ and $\mathrm{Re}\, s > \frac{w}{2} + 1$.

For Φ as in the theorem we also have explicit formulas à la Weil [B], [W] expressing $\delta(\Phi(0)+\Phi(1)) - \sum_{\rho}\Phi(p)$ as a sum over local contributions $W_{\mathfrak{p}}(F;\chi^{\sigma})$ defined using the inverse Mellin transform F of Φ. For functions Φ as in (7.16) we have in fact:

$$\mathrm{Tr}(\Phi(\Theta)|\mathcal{F}_{\mathfrak{p}}(M(\chi))_{\sigma}) = W_{\mathfrak{p}}(F;\chi^{\sigma})\ .$$

Weil's way to write explicit formulas has the advantage to apply to more general Φ than those covered by (7.16). Our way to write them makes the interpretation as a Lefschetz trace formula plausible. Perhaps the $W_{\mathfrak{p}}(F,\chi^{\sigma})$ could be interpreted as regularized traces if $\Phi(\Theta)$ is not of trace class on $\mathcal{F}_{\mathfrak{p}}(M(\chi))_{\sigma}$.

As mentioned above we expect that $\det_{\infty}\left(\frac{1}{2\pi}(s-\Theta)|H^1(\mathcal{Y},\mathcal{F}(M(\chi))\right)$ defines an entire function. Thus we are lead to consider the series

$$\xi(z,s) = \sum_{\rho}\left(\frac{1}{2\pi}(s-\rho)\right)^{-z} \tag{7.17}$$

where ρ runs over the non–trivial zeros of $L(\chi^{\sigma}, s)$. The following theorem which generalizes the result of [De2] §4 was proved by Ch. Soulé [So2] for the Riemann zeta function using the work of Cramér [Cr]. The extension to Hecke characters is due to G. Illies [I]:

(7.18) Theorem [So2], [I]: Let Ω be the set of complex numbers which are not of the form $\rho - \lambda$, $\lambda \geq 0$.

i) For any s in Ω the series for the function $\xi(z,s)$ converges absolutely if $\mathrm{Re}\, z > 1$. It extends to a meromorphic function of s in Ω and z in $\mathbb{C}$ which is regular for $z = 0$.

ii) For s in Ω we have:

$$\coprod_{\rho}\left(\frac{1}{2\pi}(s-\rho)\right) = \left(\frac{s}{2\pi}\cdot\frac{s-1}{2\pi}\right)^{\delta}\hat{L}(\chi^{\sigma}, s)$$

with δ as in (7.16).

Remark: The method of proof in [De2] §4 extends easily to Hecke characters but gives (7.18) only for $\mathrm{Re}\, s > \frac{w}{2} + 1$.

(7.19) We now wish to discuss the functional equation. Assume that for a motive M in $\mathcal{MM}_k(E)$ its $\hat{L}_S$–series is given by (7.4) and that the regularized determinants make sense for all complex s in a set Ω as above. Poincaré duality should provide a T–equivariant isomorphism between the (smooth) dual:

$$H^\nu_c(\mathcal{Y}_S, \mathcal{F}(M))^*_\sigma \;:=\; \text{direct sum of the duals of the (finite dimensional) irreducible } T\text{–subspaces in a } T\text{–invariant decomposition of } H^\nu_c(\mathcal{Y}_S, \mathcal{F}(M))_\sigma$$

and $H^{2-\nu}(\mathcal{Y}_S, \mathcal{F}(M)^*(1))_\sigma$. We get:

$$\begin{aligned}
\hat{L}_S(M,s) &= \prod_{\nu=0}^{2} \det{}_\infty \left(\frac{1}{2\pi}(s-\Theta)|H^\nu_c(\mathcal{Y}_S, \mathcal{F}(M))\right)^{(-1)^{\nu+1}} \\
&= \prod_{\nu=0}^{2} \det{}_\infty \left(\frac{1}{2\pi}(s+\Theta)|H^\nu_c(\mathcal{Y}_S, \mathcal{F}(M))^*\right)^{(-1)^{\nu+1}} \\
&= \prod_{\nu=0}^{2} \det{}_\infty \left(\frac{1}{2\pi}(s+\Theta)|H^{2-\nu}(\mathcal{Y}_S, \mathcal{F}(M)^*(1))\right)^{(-1)^{\nu+1}} \\
&= \prod_{\nu=0}^{2} \det{}_\infty \left(-\frac{1}{2\pi}((1-s)-\Theta)|H^\nu(\mathcal{Y}_S, \mathcal{F}(M^*)\right)^{(-1)^{\nu+1}} .
\end{aligned} \tag{7.19.1}$$

Now let S be empty, $\mathcal{Y}_S = \mathcal{Y}$. A standard decomposition (1.6)

$$H^\nu(\mathcal{Y}, \mathcal{F}(M)) = H^\nu(\mathcal{Y}, \mathcal{F}(M))^+ \oplus H^\nu(\mathcal{Y}, \mathcal{F}(M))^-$$

with respect to $-\Theta$ would also be standard with respect to $1-s-\Theta$ for any s. We expect that the regularized superdimension of $1-s-\Theta$

$$D_\nu(s) := \mathrm{sdim}_\infty(1-s-\Theta|H^\nu(\mathcal{Y}, \mathcal{F}(M^*))$$

exists with respect to such a decomposition c.f. (1.4), (1.6). Using (1.7) we would get:

$$\hat{L}(M,s) = \varepsilon(M,s)\hat{L}(M^*, 1-s)$$

where

$$\varepsilon(M,s) = \exp\left(i\pi \sum_{\nu=0}^{2} (-1)^{\nu+1} D_\nu(s)\right) .$$

If one could show e.g. from (7.6) that $\hat{L}(M,s)$ and $\hat{L}(M^*, 1-s)$ had genus one it would follow from the definition of the genus that

$$\varepsilon(M,s) = ae^{bs} \quad \text{for some } a \in (E \otimes \mathbb{C})^* \, , \; b \in E \otimes \mathbb{C} \, .$$

For algebraic Hecke characters there is the following result which was suggested by our considerations of regularized superdimensions:

(7.20) Theorem [I]: For s in Ω set

$$\xi^{\pm}(z,s) = \sum_{\rho}{}^{\pm} \left(\frac{1}{2\pi}(s-\rho) \right)^{-z}$$

where in $\sum_{\rho}^{\pm}$ the sum is over the non trivial zeros of $L(\chi^{\sigma}, s)$ with $\operatorname{Im}\rho \gtrless 0$. Then the sum for $\xi^{\pm}(z,s)$ converges absolutely if $\operatorname{Re} z > 1$. It extends to a meromorphic function of s in Ω and z in $\mathbb{C}$. For fixed s in Ω it has at most a first order pole at $z = 0$.

That (7.20) follows for $\chi = 1$ from results of Cramér [Cr] is evident from [So2].

(7.21) We now turn to a discussion of the order of L–functions at integral values of s. For M in $\mathcal{MM}_k(E)$ and S any finite set of places of k formula (7.19.1) gives

$$\hat{L}_S(M,s) = \prod_{\nu=0}^{2} \det{}_{\infty} \left(\frac{1}{2\pi}(s+\Theta) | H^{\nu}(\mathcal{Y}_S, \mathcal{F}(M^*(1)) \right)^{(-1)^{\nu+1}}$$

and hence

$$\operatorname{ord}_{s=0} \hat{L}_S(M,s) = \sum_{\nu=0}^{2} (-1)^{\nu+1} \dim_{\mathbb{C}} H^{\nu}(\mathcal{Y}_S, \mathcal{F}(M^*(1)))^{\Theta\sim 0}$$

in the notation of (1.1).

Now assume for the moment that $k = \mathbb{Q}$, $S = \{\infty\}$ and $E = \mathbb{Q}$ such that in particular $\hat{L}_S(M,s) = L(M,s)$. According to conjecture B of [Scho] for motives M over $\mathcal{Y}_S = \operatorname{Spec}\mathbb{Z}$ we should have:

$$\operatorname{ord}_{s=0} L(M,s) = \sum_{\nu=0}^{1} (-1)^{\nu+1} \dim_{\mathbb{Q}} \operatorname{Ext}^{\nu}_{\mathcal{MM}_{\mathbb{Z}}}(\mathbb{Q}(0), M^*(1)) \, .$$

This identity would be explained by canonical isomorphisms:

$$\operatorname{Ext}^{\nu}_{\mathcal{MM}_{\mathbb{Z}}}(\mathbb{Q}(0), M) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^{\nu}(\operatorname{Spec}\mathbb{Z}, \mathcal{F}(M))^{\Theta\sim 0} \tag{7.22}$$

for M in $\mathcal{MM}_{\mathbb{Z}}$. Replacing M by $M(n)$ gives

$$\mathrm{Ext}^{\nu}_{\mathcal{MM}_{\mathbb{Z}}}(\mathbb{Q}(0), M(n)) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^{\nu}(\mathrm{Spec}\,\mathbb{Z}, \mathcal{F}(M))^{\Theta\sim n}. \tag{7.23}$$

The expected vanishing of $\mathrm{Ext}^2_{\mathcal{MM}_{\mathbb{Z}}}$ fits in with the idea that $H^2(\mathrm{Spec}\,\mathbb{Z}, \mathcal{F}(M))$ should vanish since $\mathrm{Spec}\,\mathbb{Z}$ is "non–compact" and 1–dimensional.

The following consideration is compatible with (7.22). Let δ be the connecting morphism in the relative exact sequence for the pair $(\overline{\mathrm{Spec}\,\mathbb{Z}}, \mathrm{Spec}\,\mathbb{Z})$ where $\overline{\mathrm{Spec}\,\mathbb{Z}} = \mathrm{Spec}\,\mathbb{Z} \cup \{\infty\}$:

$$H^1(\mathrm{Spec}\,\mathbb{Z}, \mathcal{F}(M)) \xrightarrow{\delta} H^2_{\infty}(\overline{\mathrm{Spec}\,\mathbb{Z}}, \mathcal{F}(M)) \ .$$

By Poincaré duality we expect an isomorphism

$$\begin{aligned} H^2_{\infty}(\overline{\mathrm{Spec}\,\mathbb{Z}}, \mathcal{F}(M)) &\cong H^0(\infty, \mathcal{F}(M^*(1)))^* \\ &\cong \mathcal{F}_{\infty}(M^*(1))^* \ . \end{aligned} \tag{7.24}$$

In (6.8) we have shown that there is a natural map

$$\mathrm{Ext}^1_{\mathcal{MH}_{\mathbb{R}}}(\mathbb{R}(0), M_B) \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow (\mathcal{F}_{\infty}(M^*(1))^{\Theta=0})^* \tag{7.25}$$

which is an isomorphism if the weights of M are ≤ 0. It should fit into a commutative diagram

$$\begin{array}{ccccc} H^1(\mathrm{Spec}\,\mathbb{Z}, \mathcal{F}(M))^{\Theta\sim 0} & \xrightarrow{\delta} & H^2_{\infty}(\overline{\mathrm{Spec}\,\mathbb{Z}}, \mathcal{F}(M))^{\Theta\sim 0} & = & (\mathcal{F}_{\infty}(M^*(1))^{\Theta=0})^* \\ \uparrow & & & & \uparrow \\ \mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Z}}}(\mathbb{Q}(0), M) \otimes_{\mathbb{Q}} \mathbb{C} & & \xrightarrow{\text{canonical}} & & \mathrm{Ext}^1_{\mathcal{MH}_{\mathbb{R}}}(\mathbb{R}(0), M_B) \otimes_{\mathbb{R}} \mathbb{C} \ . \end{array}$$

It was this argument by the way which lead to the map (7.25) in (6.8) and thus helped to understand the relation between $\mathcal{F}_{\infty}^{\Theta=m}$ and Deligne cohomology in (6.10).

The fact that the regulator map should be viewed as a boundary map using Arakelov compactification is due to Beilinson e.g. [Be] 0.3. However the relation between his exact sequence of topological groups with volume forms and the above exact sequence is not at all obvious to me.

Note that we cannot have a Gysin isomorphism

$$H^2_{\infty}(\overline{\mathrm{Spec}\,\mathbb{Z}}, \mathcal{F}(M)) \cong H^0(\infty, \mathcal{F}(M)(-1)) = \mathcal{F}_{\infty}(M)(-1)$$

because in connection with (7.24) it would imply that $\mathcal{F}_{\infty}$ commutes with duals which it does not: e.g. Θ has weights $\ldots - 8, -4, 0$ on $\mathcal{F}_{\infty}(\mathbb{Q}(0)) = \mathbb{L}_{\infty}$ whereas the weights on

$\mathcal{F}_\infty(\mathbb{Q}(0))^*$ are $0, 4, 8, \ldots$. Again this supports the philosophy that ∞ is a singular point of $\overline{\operatorname{Spec}\mathbb{Z}}$ in the geometry of cat $\mathcal{S}$. However we expect $\overline{\operatorname{Spec}\mathbb{Z}}$ to be "smooth compact" since its cohomologies should be pure and satisfy Poincaré duality. Classically regular curves can not have singular points of course. One could also imagine that $\overline{\operatorname{Spec}\mathbb{Z}}$ is singular such that the $H^i(\overline{\operatorname{Spec}\mathbb{Z}}, j_*\mathcal{F}(M))$ would be intersection cohomology groups. In this case we would not expect that $j_*\mathcal{F}(\mathbb{Q}(0)) = \mathcal{C}_{\overline{\operatorname{Spec}\mathbb{Z}}}$.

Let X be smooth and proper over $\mathbb{Q}$ and consider the motive $M = H^i(X)(n)$ in $\mathcal{MM}_{\mathbb{Z}}$. Conjecturally the Ext–groups $\operatorname{Ext}^\nu_{\mathcal{MM}_{\mathbb{Z}}}(\mathbb{Q}(0), H^i(X)(n))$ can then be expressed in terms of Chow– and algebraic K–groups [Scho] III. In conjunction with (7.22) we are thus lead to expect the following isomorphisms:

$$
(7.26)\qquad
\begin{aligned}
H^0(\operatorname{Spec}\mathbb{Z}, \mathcal{F}(H^i(X)))^{\Theta\sim n} &= \begin{cases} (CH^n(X)/CH^n(X)^0)\otimes\mathbb{C} & \text{if } i = 2n \\ 0 & \text{if } i \neq 2n \end{cases} \\
H^1(\operatorname{Spec}\mathbb{Z}, \mathcal{F}(H^i(X)))^{\Theta\sim n} &= \begin{cases} CH^n(X)^0\otimes\mathbb{C} & \text{if } i+1 = 2n \\ H^{i+1}_{\mathcal{M}}(X,\mathbb{Q}(n))_{\mathbb{Z}}\otimes\mathbb{C} & \text{if } i+1 \neq 2n\ . \end{cases}
\end{aligned}
$$

Here $CH^n(X)^0$ is the subgroup of classes of cycles in $CH^n(X)$ which are homologically equivalent to zero and $H^k_{\mathcal{M}}(X,\mathbb{Q}(n))_{\mathbb{Z}}$ is the image of the K–theory of a regular model proper and flat over $\mathbb{Z}$ for X in $H^k_{\mathcal{M}}(X,\mathbb{Q}(n)) := \operatorname{Gr}^n_\gamma K_{2n-k}(X)_{\mathbb{Q}} \cong K^{(n)}_{2n-k}(X)_{\mathbb{Q}}$.

Extrapolating (7.26) we are lead to think of the groups $H^1(\operatorname{Spec}\mathbb{Z}, \mathcal{F}(H^i(X)))^{\Theta\sim\alpha}$ for arbitrary α in $\mathbb{C}$ as something like $H^{i+1}_{\mathcal{M}}(X,\mathbb{C}(\alpha))_{\mathbb{Z}}$ i.e. "K–theory indexed by complex numbers" an idea proposed by N. Kurokawa.

Let us return to the general case and assume that M is a mixed motive over $\mathcal{Y}_S$ with coefficients in E. For $S \supset \{\mathfrak{p}|\infty\}$ we expect isomorphisms of $E\otimes\mathbb{C}$–modules

$$
(7.27)\qquad \operatorname{Ext}^\nu_{\mathcal{MM}_{\mathcal{Y}_S}(E)}(E(0), M)\otimes_{\mathbb{Q}}\mathbb{C} \xrightarrow{\sim} H^\nu(\mathcal{Y}_S, \mathcal{F}(M))^{\Theta\sim 0}
$$

and in particular taking the limit over all finite $S \supset \{\mathfrak{p}|\infty\}$:

$$
\operatorname{Ext}^\nu_{\mathcal{MM}_k(E)}(E(0), M)\otimes_{\mathbb{Q}}\mathbb{C} \xrightarrow{\sim} H^\nu(\operatorname{Spec} k, \mathcal{F}(M))^{\Theta\sim 0}\ .
$$

For $k = E = \mathbb{Q}$ the left hand side is again expressed in terms of Chow– and K–groups in [Scho] III: just replace $H^{i+1}_{\mathcal{M}}(X,\mathbb{Q}(n))_{\mathbb{Z}}$ by $H^{i+1}_{\mathcal{M}}(X,\mathbb{Q}(n))$ in (7.26).

(7.28) Having discussed L–series of motives in some detail let us quickly consider Hasse

Weil zeta functions of quasi–projective schemes $X/\mathbb{Z}$

$$\zeta_X(s) = \prod_{x \in |X|} (1 - Nx^{-s})^{-1} .$$

Similarly as in (7.2) – (7.4) a Lefschetz trace formula of the form

$$\sum_{x \in |X|} \mathrm{Tr}(\Phi(\Theta)|\mathcal{C}_x) = \sum_i (-1)^i \mathrm{Tr}(\Phi(\Theta)|H^i_c(X, \mathcal{C})) \tag{7.29}$$

and the relation $\mathcal{C}_x = \mathbb{L}_{Nx}$ imply at least formally that

$$\zeta_X(s) = \prod_i \det{}_\infty \left(\frac{1}{2\pi}(s - \Theta)|H^i_c(X, \mathcal{C}) \right)^{(-1)^{i+1}} .$$

Now assume that X is regular connected of dimension d. Then Poincaré duality should give

$$H^i_c(X, \mathcal{C}(n))^* \cong H^{2d-i}(X, \mathcal{C}(d-n))$$

and we would get

$$\zeta_X(s) = \prod_{i=0}^{2d} \det{}_\infty \left(\frac{1}{2\pi}(s - d + \Theta)|H^i(X, \mathcal{C}) \right)^{(-1)^{i+1}} .$$

In particular we would have:

$$\mathrm{ord}_{s=d-n} \zeta_X(s) = \sum_{i=0}^{2d} (-1)^{i+1} \dim_{\mathbb{C}} H^i(X, \mathcal{C}(n))^{\Theta \sim 0} .$$

We expect formal analogues of Tate's conjecture:

$$H^i_{\mathcal{M}}(X, \mathbb{C}(n)) := \mathrm{Gr}\,^n_\gamma K_{2n-i}(X) \otimes \mathbb{C} \xrightarrow{\sim} H^i(X, \mathcal{C}(n))^{\Theta \sim 0}$$

and in particular that

$$H^i(X, \mathcal{C}(n))^{\Theta \sim 0} = 0 \quad \text{for } i > 2n .$$

This would give:

$$\mathrm{ord}_{s=d-n} \zeta_X(s) = \sum_{i=0}^{2n} (-1)^{i+1} \dim_{\mathbb{Q}} \, \mathrm{Gr}\,^n_\gamma (K_{2n-i}(X) \otimes \mathbb{Q})$$

a conjecture due to Soulé [So1]. Assume that there exists a compactification $\overline{X}$ of X over $\overline{\mathrm{Spec}\,\mathbb{Z}}$ with smooth projective infinite fibre $X_\infty = \overline{X} \setminus X$ and satisfying Poincaré duality

$$H^{i+1}_{X_\infty}(\overline{X}, \mathcal{C}(n)) = H^{2d-i-1}(X_\infty, \mathcal{C}(d-n))^* = \mathcal{F}_\infty(H^{2d-i-1}(X_\infty)(d-n))^* .$$

The natural map α of (6.9) which is an isomorphism for $i < 2n$ should fit into a commutative diagram

$$\begin{array}{ccccc} H^i(X,\mathcal{C}(n))^{\Theta\sim 0} & \xrightarrow{\delta} & H^{i+1}_{X_\infty}(\overline{X},\mathcal{C}(n))^{\Theta\sim 0} & = (\mathcal{F}_\infty(H^{2d-i-1}(X_\infty)(d-n))^{\Theta=0})^* \\ \wr\uparrow & & & \alpha\uparrow \\ H^i_{\mathcal{M}}(X,\mathbb{Q}(n))\otimes_{\mathbb{Q}}\mathbb{C} & & \xrightarrow{\quad r_{\mathcal{D}}\quad} & H^i_{\mathcal{D}}(X_\infty,\mathbb{R}(n))\otimes_{\mathbb{R}}\mathbb{C} \end{array}$$

where $r_{\mathcal{D}}$ is the regulator map.

(7.30) It is not clear to me how the Lefschetz trace formula should look like if we replace X by $\overline{X}$ in (7.29). What are the "fixed points under Θ" at infinity? The connection between the values of L–functions at integral points and volumes conjectured by Beilinson, Bloch–Kato et al. still remains mysterious even at our formal level. Note however that in theoretical physics integrals over path spaces are often defined formally by regularized determinants of certain operators [Wi1] §3. Hence there should be a relation between measure theory on infinite dimensional spaces and such determinants. I expect such a theory to be necessary for a proof of the conjectures on special values of L-series.

(7.31) Let $CH^1(\overline{\mathrm{Spec}\,\mathbb{Z}})$ denote the first Arakelov Chow group of $\overline{\mathrm{Spec}\,\mathbb{Z}}$. I expect a cycle class map such that composition with the trace map:

$$CH^1(\overline{\mathrm{Spec}\,\mathbb{Z}}) \longrightarrow H^2(\overline{\mathrm{Spec}\,\mathbb{Z}},\mathcal{C}(1))^{\Theta=0} \xrightarrow[\sim]{\mathrm{Tr}} \mathbb{C}$$

equals the Arakelov degree. Since the latter surjects onto $\mathbb{R}$ we are somewhat confirmed in our basic assumption that arithmetic cohomology should have coefficients in a field C at least containing $\mathbb{R}$. The local constructions in section 3 render $C = \mathbb{R}$ improbable, $C = \mathbb{C}$ seems to be the minimal choice.

Of course higher dimensional Arakelov Chow groups should have cycle maps into arithmetic cohomology but we will not discuss this here. No doubt the ideas of Gillet and Soulé in [Gi–So] will fit into the picture.

(7.32) We close this section with some remarks about a Knneth formula for arithmetic cohomology. The category cat $\mathcal{S}$ should have a product $\underline{\times}$ such that if X and Y are schemes $X \underline{\times} Y \neq X \times_{\mathrm{Spec}\,\mathbb{Z}} Y$ is no longer a scheme in general. Since Θ comes from a Lie algebra representation a Knneth formula

$$H_c^\bullet(X\underline{\times}Y,\mathcal{C}) \cong H_c^\bullet(X,\mathcal{C}) \otimes_{\mathbb{C}} H_c^\bullet(Y,\mathcal{C})$$

would imply that Θ on the left corresponds to $\Theta \otimes \mathrm{id} + \mathrm{id} \otimes \Theta$ on the right. In particular we would have:

$$\det_\infty\left(\frac{1}{2\pi}(s-\Theta)|H^2(\overline{\operatorname{Spec}\mathbb{Z}}\underline{\times}\overline{\operatorname{Spec}\mathbb{Z}},\mathcal{C})\right) = \left(\frac{s-1}{2\pi}\right)^2 \prod_{\rho,\rho'} \frac{1}{2\pi}(s-(\rho+\rho'))$$

where ρ and ρ' run over the non–trivial zeros of $\zeta(s)$. Of course the regularized product on the right does not make sense. However similar products but with the restriction $\operatorname{Im}\rho, \operatorname{Im}\rho' \geq 0$ or $\operatorname{Im}\rho, \operatorname{Im}\rho' < 0$ have been proposed by N. Kurokawa [K] as zeta–functions of $\overline{\operatorname{Spec}\mathbb{Z}} \times \overline{\operatorname{Spec}\mathbb{Z}}$. His functions are of order two and there is hope that they have similar arithmetic properties as classical L–functions. Let us illustrate this by an example where no convergence problems arise: Let K be $\mathbb{R}$ or $\mathbb{C}$ with unique place $\mathfrak{p}$ set $e_\mathfrak{p} = [\mathbb{C} : K]$ and let X be an algebraic scheme over $\mathbb{Z}$. Then the zeta–function of $X\underline{\times}\mathfrak{p}$ should be given by the infinite product

$$\zeta_{X\underline{\times}\mathfrak{p}}(s) := \prod_{\nu=0}^{\infty} \zeta_X(s+\nu e_\mathfrak{p}) \tag{7.33}$$

which converges absolutely to an analytic function in the region where ζ_X is analytic: note that the Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ for $\zeta_X(s)$ has $a_1 = 1$ as its first coefficient and hence that $\zeta_X(s) \to 1$ strongly for $\operatorname{Re} s \to \infty$. The following formal computation justifies the definition (7.33):

$$\begin{aligned}
\zeta_{X\underline{\times}\mathfrak{p}}(s) &= \prod_i \det_\infty\left(\frac{1}{2\pi}(s-\Theta)|H_c^i(X\underline{\times}\mathfrak{p},\mathcal{C})\right)^{(-1)^{i+1}} \\
&= \prod_i \det_\infty\left(\frac{1}{2\pi}(s-\Theta)|H_c^i(X,\mathcal{C}) \otimes_{\mathbb{C}} \mathbb{L}_\mathfrak{p}\right)^{(-1)^{i+1}} \\
&= \prod_i \prod_{\nu=0}^{\infty} \det_\infty\left(\frac{1}{2\pi}(s-\Theta)|H_c^i(X,\mathcal{C})(\nu\, e_\mathfrak{p})\right)^{(-1)^{i+1}} \\
&= \prod_i \prod_{\nu=0}^{\infty} \det_\infty\left(\frac{1}{2\pi}(s+\nu\, e_\mathfrak{p}-\Theta)|H_c^i(X,\mathcal{C})\right)^{(-1)^{i+1}} \\
&= \prod_{\nu=0}^{\infty} \zeta_X(s+\nu\, e_\mathfrak{p})\ .
\end{aligned}$$

Clearly $\zeta_{X \underline{\times} \mathfrak{p}}(s)$ has the Euler product $\prod_p \zeta_{X_p \underline{\times} \mathfrak{p}}(s)$ where

$$\zeta_{X_p \underline{\times} \mathfrak{p}}(s) = \prod_{\nu=0}^{\infty} \zeta_{X_p}(s + \nu e_{\mathfrak{p}}) \quad , \quad X_p = X \otimes_{\mathbb{Z}} \mathbb{F}_p$$

and it can be written as a Dirichlet series. The case where $X = \operatorname{Spec} \mathcal{O}_k$ with k a finite extension of $\mathbb{Q}$ is considered in detail by Cohen and Lenstra [C–L] in their heuristic study of class groups of number fields as was kindly pointed out to me by Manin. They prove a functional equation for $\zeta_{X \underline{\times} \mathfrak{p}}(s)$ in loc. cit. Theorem 7.1 which would also be explained by Poincaré duality on X. If $K = \mathbb{C}$ and $X = \operatorname{Spec} \mathbb{Z}$ their corollary (3.7) specializes to the formula:

$$\begin{aligned} \zeta_{\operatorname{Spec} \mathbb{Z} \underline{\times} \mathfrak{p}}(s) &= \prod_{\nu=0}^{\infty} \zeta(s+\nu) \\ &= \sum_{\mathcal{A}} \frac{|\mathcal{A}|}{|\operatorname{Aut} \mathcal{A}|} \frac{1}{|\mathcal{A}|^s} \quad , \ \operatorname{Re} s > 1 \ . \end{aligned}$$

Here $\mathcal{A}$ runs over all finite abelian groups (i.e. coherent torsion sheaves on $\operatorname{Spec} \mathbb{Z}$) up to isomorphism.

(7.34) In order to carry out the program of §7 the most important step is of course to find the objects and morphisms of the category cat $\mathcal{S}$ and to study the new geometry that they give rise to. Foundations again!

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