Motivic L–functions and regularized determinants II

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0. Introduction

In previous papers [D1, D3] we tried to shed some light on the possibility of a global cohomological understanding of the L–functions of motives over number fields. A very satisfactory picture emerged in that many standard conjectures on such L–series would be explained by natural properties of infinite dimensional cohomologies. In addition the as yet speculative cohomological approach suggested a number of consequences that could be proved by classical methods [D2, D4], [D–N], [S].

To get the cohomological approach started it appears necessary to discover the underlying geometry and a suitable site. This seems to be a difficult problem. The H^1 of the space associated to $\operatorname{spec} \mathbb{Z} \cup \{\infty\}$ in this new geometry for example would carry a canonical automorphism whose eigenvalues would be the non–trivial zeros of the Riemann zeta function. Mathematicians have been looking for such an operator since the time of Hilbert, but no really satisfactory construction has been found. Without candidates for H^1's or π_1's as guiding objects it is very hard to imagine the nature of the sought for geometry. When the étale and the crystalline sites were discovered at least the Tate– and the Dieudonne module were already known. Essentially the only guidelines we have come from interpreting assertions in analytic number theory (of motives) in cohomological terms. This will then imply properties of the looked for cohomology theory. The hints we get from these properties concerning the nature of the "spaces" we are investigating in dealing with $\operatorname{spec} \mathbb{Z} \cup \{\infty\}$ for example remain very scarse.

In my opinion the evidence accumulated so far and the striking compatibility with proven and conjectured properties of motivic L–series can only be explained by the existence of such spaces and their cohomology.

The aim of this note is mainly to continue the discussion of [D1] § 7 on the relations between the prospected cohomology and properties of Ext–groups and L–series of motives over number fields. As yet this is the only way to obtain some information on these cohomologies. I should point out that rather little is proved and even the assertions often involve quantities for which there is as yet no rigorous definition. Therefore originally I did not intend to publish these ideas. However I changed my mind in view of the fantastic impact the searched for cohomology theory would have in arithmetic geometry and the help these notes might possibly lend to its construction. I would like to thank the Institute for Advanced Study in Princeton for a pleasant stay during which some work on this project was done. I would also like to thank the referee for helpful remarks.

1. Some preliminary comments on [D1] § 7

We assume the contents and the notations of [D1]. However we are more cautious than in loc. cit. expecting only an action of the semigroup $T^+ = (\mathbb{R}^{\geq 0}, +)$ on the objects of the arithmetic site and not an action of $(\mathbb{R}, +)$. One reason for this comes from the analogy with the Frobenius semigroup action below. Another reason is that explicit formulas in analytic number theory take the form of a Lefschetz trace formula only if we restrict attention to test functions with support on $\mathbb{R}^+$ c.f. [D–Schr] (3.2.3) 1). The operator Θ will denote the infinitesimal generator of the induced semigroup action on sheaves and cohomology. The symbol $\mathcal{Y} = \mathcal{Y}_k$ will signify the "space" associated to $\operatorname{spec}(\mathcal{O}_k) \cup \{p \mid \infty\}$ for a number field $k/\mathbb{Q}$. It should be considered as analogous to $Y = Y_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ for a proper curve $Y_0/\mathbb{F}_q$, the semigroup action of T^+ on $\mathcal{Y}$ corresponding to the action of the semigroup $\mathbb{Z}^+ \cong \{\mathrm{Fr}_q^\nu \mid \nu \geq 0\}$. The space $\mathcal{Y}$ is expected to be highly singular in the points corresponding to the archimedean places of k. So in our analogy Y should have singularities. Let K_0 be the function field of Y_0 and let $j_0 : \operatorname{spec} K_0 \to Y_0$ resp. $\tilde{j}_0 : \operatorname{spec} K \to \tilde{Y}_0$ denote the inclusions of the generic point into Y_0 resp. its normalization $\tilde{Y}_0$. Then for any smooth E_λ–sheaf $\mathcal{M}_0$ on $\operatorname{spec} K_0$ we have

$$H^i(Y, j_* \mathcal{M}) = H^i(\tilde{Y}, \tilde{j}_* \mathcal{M}) \tag{1.1}$$

since $\pi : \tilde{Y} \to Y$ is finite. Here as usual omitting the index "0" means base extension to $\overline{\mathbb{F}}_q$. From (1.1) it is clear that Poincaré duality [De1] Th. (2.12) and the main result on weights in [De2] Th. (3.2.3) remain valid for singular curves. The deeper reason is that $H^i(Y, j_* \mathcal{M})$ can also be viewed as intersection cohomology.

In [D1] for every mixed motive M over k with coefficients in a number field E we have constructed certain $\mathbb{C}$–vector spaces with T^+–action $\mathcal{F}_p(M)$ (in fact with a $T = \mathbb{R}$–action) for every place p of k. We had postulated that they were stalks of a sheaf $\mathcal{F}(M)$ of $E \otimes \mathbb{C}$–modules with T^+–action on $\mathcal{Y}$, i.e.

$$\mathcal{F}(M)_p = \mathcal{F}_p(M) \, . \tag{1.2}$$

The sheaf $\mathcal{F}(M)$ was considered analogous to $j_* \mathcal{M}$ for a mixed E_λ–sheaf $\mathcal{M}_0$ on $\operatorname{spec} K_0$. From the analogy with (singular) curves over finite fields we expect Poincaré duality and a theory of weights also for the hypothetical cohomology groups $H^i(\mathcal{Y}, \mathcal{F}(M))$. This and the consequences for L–series of motives were already explained in [D1] § 7. The expected structure sheaf of $\mathbb{C}$–algebras with T^+–action on $\mathcal{Y}$ was denoted by $\mathcal{C}$ in loc. cit. Since $\mathcal{Y}$ is singular we do not expect that $j_* \mathcal{C}_{\operatorname{spec} k} = \mathcal{C}_{\mathcal{Y}}$ where $j : \operatorname{spec} k \to \mathcal{Y}$ is the inclusion i.e. we expect that $\mathcal{F}(\mathbb{Q}(0)) \neq \mathcal{C}_{\mathcal{Y}}$. Note the discussion between formulas (7.25) and (7.26) in [D1]. In the discussion of the zeta function of k in loc. cit. the groups $H^i(\mathcal{Y}, \mathcal{F}(\mathbb{Q}(0)))$ appear. There we also wrote $H^i(\mathcal{Y}, \mathcal{C})$ for them. However in the latter group $\mathcal{Y}$ should rather be replaced by some kind of "normalization".

Let us recall for further reference what the Dedekind zeta function tells us about the groups $H^i(\mathcal{Y}, \mathcal{F}(\mathbb{Q}(0))$. By [D1] (7.9), (7.10) we have

$$H^0(\mathcal{Y}, \mathcal{F}(\mathbb{Q}(0))) = \mathbb{C}(0) \qquad H^2(\mathcal{Y}, \mathcal{F}(\mathbb{Q}(0))) \overset{\mathrm{Tr}}{\cong} \mathbb{C}(-1)$$

where the infinitesimal generator Θ of the T^+–action acts by $\Theta_{\mathbb{C}(s)} = -s\,\mathrm{id}$ on the one dimensional $\mathbb{C}$–vector space $\mathbb{C}(s)$. Moreover $H^1(\mathcal{Y}, \mathcal{F}(\mathbb{Q}(0)))$ decomposes into the finite dimensional generalized eigenspaces $\varinjlim_N \mathrm{Ker}\,(\Theta - \rho\,\mathrm{id})^N$ of Θ with eigenvalues ρ the non–trivial zeros of $\zeta_k(s)$ with their multiplicities.

2. Ext–groups of motives, K–theory and the prospected cohomologies

Let S be a finite set of places of k and set $\mathcal{Y}_S = \mathcal{Y} \setminus S$. The category of motives $\mathcal{MM}_{\mathcal{Y}_S}$ is by definition the full subcategory of $\mathcal{MM}_k$ of motives which are integral at all places $p \notin S$ in the sense of [Sch] resp. [D–N]. Each of the functors

$$\mathcal{F}_p : \mathcal{MM}_k \longrightarrow \mathcal{D}_p$$

constructed in [D1] restricts to an *exact* functor on the subcategory of motives which are integral at p. Thus for all $p \notin S$ the functors

$$\mathcal{F}_p : \mathcal{MM}_{\mathcal{Y}_S} \longrightarrow \mathcal{D}_p$$

are exact and hence by (1.2) one is lead to assume that the functor

$$\mathcal{F} : \mathcal{MM}_{\mathcal{Y}_S} \longrightarrow \text{sheaves on } \mathcal{Y}_S$$

is exact as well. It will then induce a homomorphism on ν–extension groups:

$$\mathcal{F} : \mathrm{Ext}^{\nu}_{\mathcal{MM}_{\mathcal{Y}_S}}(\mathbb{Q}(0), M) \longrightarrow \mathrm{Ext}^{\nu}_{\mathcal{Y}_S}(\mathcal{F}(\mathbb{Q}(0)), \mathcal{F}(M))^{\Theta\sim 0} \tag{2.1}$$

where $H^{\Theta\sim\alpha}$ is the subspace of H annihilated by $(\Theta - \alpha\,\mathrm{id})^N$ for some $N > 0$.

If S contains all the infinite primes the motivic Ext–groups are expected to be finite dimensional. Moreover $\mathcal{Y}_S$ would then be smooth and we should have $\mathcal{F}(\mathbb{Q}(0)) = \mathcal{C}_{\mathcal{Y}_S}$ so that

$$\mathrm{Ext}^{\nu}_{\mathcal{Y}_S}(\mathcal{F}(\mathbb{Q}(0)), \mathcal{F}(M)) = H^{\nu}(\mathcal{Y}_S, \mathcal{F}(M))\,.$$

Refining [D1] (7.27) we expect that $\mathcal{F}$ induces isomorphisms

$$\mathcal{F} : \mathrm{Ext}^{\nu}_{\mathcal{MM}_{\mathcal{Y}_S}}(\mathbb{Q}(0), M) \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow H^{\nu}(\mathcal{Y}_S, \mathcal{F}(M))^{\Theta\sim 0} \quad \text{if } S \supset \{p \,|\, \infty\} \tag{2.2}$$

and more generally for motives with coefficients in E isomorphisms of $E \otimes \mathbb{C}$–modules of finite rank

$$\mathcal{F} : \mathrm{Ext}^{\nu}_{\mathcal{MM}_{\mathcal{Y}_S(E)}}(E(0), M) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^{\nu}(\mathcal{Y}_S, \mathcal{F}(M))^{\Theta\sim 0} \quad \text{if } S \supset \{p \,|\, \infty\}\,. \tag{2.3}$$

Concerning arbitrary finite S it appears from [D–N] that the groups $\mathrm{Ext}^{\nu}(E(0), M)$ are no longer finite dimensional in general and that the process of tensoring with

$\mathbb{C}$ should be replaced by a suitable $\mathbb{C}$–completion denoted by the index $\mathbb{C}$. We expect to get an isomorphism for any S:

$$(2.4)\qquad \mathcal{F} : \mathrm{Ext}^{\nu}_{\mathcal{MM}_{\mathcal{Y}_S(E)}}(E(0), M)_{\mathbb{C}} \xrightarrow{\ \sim\ } \mathrm{Ext}^{\nu}_{\mathcal{Y}_S}(\mathcal{F}(\mathbb{Q}(0)), \mathcal{F}(M))^{\Theta\sim 0} .$$

Note that if S does not contain all infinite places we expect that $\mathcal{F}(\mathbb{Q}(0)) \neq \mathcal{C}_{\mathcal{Y}_S}$ and therefore that the Ext–group on the right hand side does not in general equal the cohomology of $\mathcal{Y}_S$. This will be used to "explain" certain assymmetries between finite and infinite places in section 5.

We close this section with a comment on the isomorphism predicted in section (7.28) of [D1]:

$$(2.5)\qquad H^i_{\mathcal{M}}(X, \mathbb{C}(n)) := \mathrm{Gr}^{\,n}_{\gamma} K_{2n-i}(X) \otimes \mathbb{C} \xrightarrow{\ \sim\ } H^i(X, \mathcal{C}(n))^{\Theta\sim 0}$$

for $X/\mathbb{Z}$ a regular, quasi–projective scheme. For X purely of characteristic p i.e. $X/\mathbb{F}_p$ we have defined a candidate for $H^i(X, \mathcal{C}(n))$ in [D1] §4 namely:

$$H^i(X, \mathcal{C}(n)) \cong H^i(X/\mathbb{L})(n)\ ,\ \ \Theta \mathrel{\hat{=}} \Theta_p$$

in the notation of loc. cit. Now according to [D1] (2.4) and the construction of $H^i(X/\mathbb{L})$ we have:

$$H^i(X/\mathbb{L})(n)^{\Theta_p\sim 0} = H^i(X/\mathbb{L})(n)^{\Theta_p = 0} \cong H^i(X \otimes \overline{\mathbb{F}}_p, \mathbb{Q}_l(n))^{\mathrm{Fr}_p} \otimes_{\mathbb{Q}_l,\iota} \mathbb{C}$$

for a fixed embedding $\iota : \mathbb{Q}_l \hookrightarrow \mathbb{C}$. Thus in this case (2.5) is equivalent to Tate's conjecture [J] §12

$$H^i_{\mathcal{M}}(X, \mathbb{Q}_l(n)) := \mathrm{Gr}^{\,n}_{\gamma} K_{2n-i}(X) \otimes \mathbb{Q}_l \xrightarrow{\ \sim\ } H^i(X \otimes \overline{\mathbb{F}}_p, \mathbb{Q}_l(n))^{\mathrm{Fr}_p} .$$

3. The Artin conjecture for motives

For a motive M in $\mathcal{MM}_k(E)$ its L–series $L(M,s)$ is expected to have a meromorphic continuation to $\mathbb{C}$. The Artin conjecture says that $L(M,s)$ is actually everywhere holomorphic except if M admits a non–trivial map $M \to E(\nu)$ for some $\nu \in \mathbb{Z}$. This would be explained as follows using the equality of $E \otimes \mathbb{C} = \mathbb{C}^{\mathrm{Hom}(E,\mathbb{C})}$–valued functions in [D1] (7.21):

$$(3.1)\qquad L(M,s) = \prod_{\nu=0}^{1} \mathrm{det}_{\infty}\left(\frac{1}{2\pi}(s+\Theta)\,|\, H^{\nu}(\text{“}\mathrm{spec}\,\mathcal{O}_k\text{”}, \mathcal{F}(M^*(1))\right)^{(-1)^{\nu+1}}$$

where "$\mathrm{spec}\,\mathcal{O}_k$" is the space associated to $\mathrm{spec}\,\mathcal{O}_k$, if we assume that the only eigenvalues of Θ on $H^0(\text{“}\mathrm{spec}\,\mathcal{O}_k\text{”}, \mathcal{F}(N))$ for N in $\mathcal{MM}_k(E)$ are integers. In this case we would have

$$(3.2)\qquad H^0(\text{“}\mathrm{spec}\,\mathcal{O}_k\text{”}, \mathcal{F}(N)) = \bigoplus_{n\in\mathbb{Z}} H^0(\text{“}\mathrm{spec}\,\mathcal{O}_k\text{”}, \mathcal{F}(N))^{\Theta\sim n} .$$

For every finite set of places S we expect the equality

$$H^0(\mathcal{Y}, \mathcal{F}(N)) = H^0(\mathcal{Y}_S, \mathcal{F}(N)) = H^0(\text{“spec}\, k\text{”}, \mathcal{F}(N)) \tag{3.3}$$

since Poincaré duality would give

$$\begin{aligned} H^\nu_S(\mathcal{Y}, \mathcal{F}(N)) &= \bigoplus_{p\in S} H^{2-\nu}(p, \mathcal{F}(N^*(1)))^* \\ &= 0 \quad \text{for } \nu = 0, 1\,. \end{aligned} \tag{3.4}$$

Note that e.g. in l–adic cohomology over algebraically closed fields the corresponding duality is also true for singular curves.

In particular combining (2.3) and (3.3) we should have

$$H^0(\text{“spec}\, \mathcal{O}_k\text{”}, \mathcal{F}(N))^{\Theta\sim n} \cong \mathrm{Hom}_{\mathcal{MM}_k(E)}(E(0), N(n)) \otimes_{\mathbb{Q}} \mathbb{C}\,. \tag{3.5}$$

Together with (3.2) and (3.1) the Artin conjecture would follow. Note that because of (3.5) the integer n can be an eigenvalue of Θ on $H^0(\text{“spec}\, \mathcal{O}_k\text{”}, \mathcal{F}(N)) = H^0(\mathcal{Y}, \mathcal{F}(N))$ only if $2n$ is among the weights of N. This is compatible with the following expected theory of weights on our cohomologies c.f. [D1] (7.5):

(3.6) Weights: The real parts of the eigenvalues of Θ on $H^\nu(\mathcal{Y}, \mathcal{F}(N))$ are of the form $\frac{w+\nu}{2}$ for some weight w of N.

Returning to (3.1) we see more precisely that $L(M, s)$ can have a pole at s only if $s = n$ is an integer such that there is a non–trivial map $M \to E(1-n)$. In particular $2n - 2$ must then be a weight of M.

(3.7) Since Θ is a derivation with respect to cup product on the global cohomologies Kontsevich and Manin as I understand it suggested to look for additive relations among the zeroes of the Hasse–Weil zeta functions (of integral models) of X, Y and $X \times Y$ for varieties X, Y over $\mathbb{Q}$. No non–trivial such relations were found by the computer in examples though. In fact this is not surprising since the $\cup$–product gives pairings:

$$H^i(\mathcal{Y}, \mathcal{F}(H^p(X))) \otimes H^j(\mathcal{Y}, \mathcal{F}(H^q(Y))) \longrightarrow H^{i+j}(\mathcal{Y}, \mathcal{F}(H^p(X) \otimes H^q(Y)))$$

and (3.2), (3.3) imply that the only eigenvalues of Θ on $H^0(\mathcal{Y}, \mathcal{F}(M))$ and

$$H^2(\mathcal{Y}, \mathcal{F}(M)) \cong H^0(\mathcal{Y}, \mathcal{F}(M^*(1)))^*$$

are integers. Thus using the formula [D1] (7.4) for the completed L–series of any M in $\mathcal{MM}_k(E)$:

$$\hat{L}(M, s) = \prod_{i=0}^{2} \det_\infty \left(\frac{1}{2\pi}(s - \Theta) \,|\, H^i(\mathcal{Y}, \mathcal{F}(M)) \right)^{(-1)^{i+1}} \tag{3.8}$$

we conclude in particular: The derivation property of Θ implies at most relations of the form: $\rho_1 + \rho_2 = \rho_3$ between zeroes and poles of $\hat{L}(H^p(X), s)$, $\hat{L}(H^q(Y), s)$ and $\hat{L}(H^{p+q}(X \times Y), s)$ where at least one of the ρ_i is a pole and hence an integer.

The deeper reason why we don't get interesting relations between three zeroes in this way is of course this: the fibre products $X \times Y$ are taken over the generic point $\operatorname{spec}\mathbb{Q}$ of the curve $\mathcal{Y}$ and not over an absolute "base point".

4. "Explanations" for conjectures of Deligne and Scholl on critical motives

For a motive M in $\mathcal{MM}_{\mathbb{Q}}(E)$ consider the composite map $I_\infty^+(M)$:

$$M_B^+ \otimes \mathbb{R} \hookrightarrow M_B \otimes \mathbb{C} \xrightarrow{\sim} M_{DR} \otimes \mathbb{C} \twoheadrightarrow M_{DR} \otimes \mathbb{R} \twoheadrightarrow \frac{M_{DR}}{F^0} \otimes \mathbb{R}\,.$$

In [De3], [Sch] the motive is called critical if $I_\infty^+(M)$ is an isomorphism. It is easy to see that this condition is equivalent to the L–factors at infinity $L_\infty(M,s)$ and $L_\infty(M^*(1),s)$ not having a pole at $s=0$. A conceptual proof runs as follows: Because of formula [D1] (6.5)

$$L_\infty(M,s) = \det\nolimits_\infty \left(\frac{1}{2\pi}(s-\Theta) \,|\, \mathcal{F}_\infty^{\mathbb{R}}(M) \right)^{-1}$$

where $\mathcal{F}_\infty^{\mathbb{R}}(M)$ is a natural $\mathbb{R}$–structure on $\mathcal{F}_\infty(M)$ introduced in loc. cit. the L–factors do not have a pole at $s=0$ if and only if

$$\text{(4.1)} \qquad \mathcal{F}_\infty^{\mathbb{R}}(M)^{\Theta=0} = 0 \quad \text{and} \quad \mathcal{F}_\infty^{\mathbb{R}}(M^*(1))^{\Theta=0} = 0\,.$$

We have

$$\mathcal{F}_\infty^{\mathbb{R}}(M)^{\Theta=0} = (F^0 M_{DR} \otimes \mathbb{C} \cap M_B \otimes \mathbb{R})^+ = \operatorname{Ker}(I_\infty^+(M))$$

and

$$\mathcal{F}_\infty^{\mathbb{R}}(M^*(1))^{\Theta=0} = \operatorname{Ker}(I_\infty^+(M^*(1))) \cong (\operatorname{Coker}(I_\infty^+(M)))^*$$

by [De3] § 5.1 and the claim follows.

For N in $\mathcal{MM}_{\mathbb{Q}}(E)$ the group

$$H_\infty^\nu(\mathcal{Y}, \mathcal{F}(N)) = H^{2-\nu}(\infty, \mathcal{F}(N^*(1)))^*$$

is zero for $\nu = 0,1$ and isomorphic to $\mathcal{F}_\infty(N^*(1))^*$ for $\nu = 2$. Moreover $H^2(\text{“spec}\,\mathbb{Z}\text{”}, \mathcal{F}(N)) = 0$ since $\operatorname{spec}\mathbb{Z}$ is an affine curve.

Thus the relative exact sequence for the pair "$\operatorname{spec}\mathbb{Z}$" $\subset \mathcal{Y}$ gives the exact sequence:

$$0 \to H^1(\mathcal{Y}, \mathcal{F}(N)) \to H^1(\text{“spec}\,\mathbb{Z}\text{”}, \mathcal{F}(N)) \to \mathcal{F}_\infty(N^*(1))^* \to H^2(\mathcal{Y}, \mathcal{F}(N)) \to 0$$

and the isomorphism (c.f. (3.3))

$$H^0(\mathcal{Y}, \mathcal{F}(N)) = H^0(\text{“spec}\,\mathbb{Z}\text{”}, \mathcal{F}(N))\,.$$

Thus M is critical if and only if for $N = M, M^*(1)$ the injection

$$H^1(\mathcal{Y}, \mathcal{F}(N))^{\Theta\sim 0} \to H^1(\text{“spec}\,\mathbb{Z}\text{”}, \mathcal{F}(N))^{\Theta\sim 0} \tag{4.2}$$

is an isomorphism and

$$H^0(\text{“spec}\,\mathbb{Z}\text{”}, \mathcal{F}(N))^{\Theta\sim 0} = 0\,. \tag{4.3}$$

Concerning (4.3) note that by Poincaré duality [D1] (7.19) we have

$$H^0(\text{“spec}\,\mathbb{Z}\text{”}, \mathcal{F}(N)) = H^0(\mathcal{Y}, \mathcal{F}(N)) = H^2(\mathcal{Y}, \mathcal{F}(N^*(1)))^*\,.$$

Using (3.1) we therefore see that if M is critical $L(M, s)$ does not have a pole at $s = 0$. Moreover in this case it follows from (3.6) that $L(M, 0) \neq 0$ if $w = -1$ is not a weight of M. In the pure case this was conjectured by Deligne [De3] (2.7) (i). Note also that conjecture (2.7) (ii) of loc. cit. follows from (2.3).

By duality, conditions (4.2) and (4.3) for $N = M^*(1)$ are equivalent to the conditions that the surjection

$$H^1_c(\text{“spec}\,\mathbb{Z}\text{”}, \mathcal{F}(M))^{\Theta\sim 0} \longrightarrow H^1(\mathcal{Y}, \mathcal{F}(M))^{\Theta\sim 0} \tag{4.4}$$

is an isomorphism and that

$$H^2_c(\text{“spec}\,\mathbb{Z}\text{”}, \mathcal{F}(M))^{\Theta\sim 0} = 0\,. \tag{4.5}$$

Using this and the factorization

$$\begin{tikzcd} & H^1(\mathcal{Y}, \mathcal{F}(M)) \arrow[dr, hook] & \\ H^1_c(\text{“spec}\,\mathbb{Z}\text{”}, \mathcal{F}(M)) \arrow[ur, two heads] \arrow[rr] & & H^1(\text{“spec}\,\mathbb{Z}\text{”}, \mathcal{F}(M)) \end{tikzcd}$$

we see that M is critical if and only if the map

$$H^1_c(\text{“spec}\,\mathbb{Z}\text{”}, \mathcal{F}(M))^{\Theta\sim 0} \longrightarrow H^1(\text{“spec}\,\mathbb{Z}\text{”}, \mathcal{F}(M))^{\Theta\sim 0} \tag{4.6}$$

is an isomorphism and if

$$H^0(\text{“spec}\,\mathbb{Z}\text{”}, \mathcal{F}(M))^{\Theta\sim 0} = 0 = H^2_c(\text{“spec}\,\mathbb{Z}\text{”}, \mathcal{F}(M))^{\Theta\sim 0}\,. \tag{4.7}$$

Via the isomorphism (2.2) and Poincaré duality Scholl's condition for a motive M in $\mathcal{MM}_{\mathbb{Z}}(E)$ to be highly criticial [Sch] III:

$$\mathrm{Ext}^i_{\mathcal{MM}_{\mathbb{Z}}(E)}(E(0), N) = 0 \quad \text{for } N = M, M^*(1) \text{ and } i = 0, 1 \tag{4.8}$$

translates into the conditions

$$H^1_c(\text{“spec}\,\mathbb{Z}\text{”}, \mathcal{F}(M))^{\Theta\sim 0} = 0 = H^1(\text{“spec}\,\mathbb{Z}\text{”}, \mathcal{F}(M))^{\Theta\sim 0} \tag{4.9}$$

and

$$H^0(\text{“spec}\,\mathbb{Z}\text{”},\mathcal{F}(M))^{\Theta\sim 0}=0=H^2_c(\text{“spec}\,\mathbb{Z}\text{”},\mathcal{F}(M))^{\Theta\sim 0}\,. \tag{4.10}$$

In particular a motive over $\mathbb{Z}$ will be critical if it is highly critical – a conjecture of Scholl [Sch] III, Conj. C.

All considerations in this section extend to motives over a number field.

Remark: Assume that M in $\mathcal{MM}_{\mathbb{Q}}(E)$ is critical. According to Deligne [De3] and Scholl [Sch] it is expected that

$$L(M,0)=c\det{}_{\mathbb{Q}}I^+_\infty(M)\quad\text{for some } c \text{ in } \mathbb{Q}\,. \tag{4.11}$$

Assuming the motivic formalism this conjecture is equivalent to the Beilinson conjectures. According to §§ 2, 3 we would have in $E\otimes\mathbb{C}=\mathbb{C}^{\mathrm{Hom}(E,\mathbb{C})}$:

$$\begin{aligned} L(M,0) &= \prod_{\nu=0}^{1}\det{}_\infty\left(\frac{\Theta}{2\pi}\mid H^\nu(\text{“spec}\,\mathbb{Z}\text{”},\mathcal{F}(M^*(1))\right)^{(-1)^{\nu+1}}\\ &= \prod_{n\in\mathbb{Z}}\left(\frac{2\pi}{n}\right)^{a_n}\det{}_\infty\left(\frac{\Theta}{2\pi}\mid H^1(\text{“spec}\,\mathbb{Z}\text{”},\mathcal{F}(M^*(1))\right)\end{aligned}$$

where

$$a_n=\dim_E\mathrm{Hom}_{\mathcal{MM}_{\mathbb{Q}}(E)}(M,E(n+1))\,.$$

A principle relating

$$\begin{aligned}\det{}_\infty\left(\frac{\Theta}{2\pi}\mid H^1(\text{“spec}\,\mathbb{Z}\text{”},\mathcal{F}(M^*(1))\right) &= \det{}_\infty\left(-\frac{\Theta}{2\pi}\mid H^1_c(\text{“spec}\,\mathbb{Z}\text{”},\mathcal{F}(M))\right)\\ &\overset{(4.6)}{=} \det{}_\infty\left(-\frac{\Theta}{2\pi}\mid H^1(\text{“spec}\,\mathbb{Z}\text{”},\mathcal{F}(M))\right)\end{aligned}$$

to the finite determinant $\det_{\mathbb{Q}} I^+_\infty(M)$ up to powers of 2π and rational factors remains to be found. The phenomenon that zeta–regularized determinants on infinite dimensional spaces are sometimes expressible by finite dimensional determinants in a natural way has been observed before e.g. in physics c.f. [F]. However the mechanism remains unclear. How do the $\mathbb{Q}$–structures on M^+_B and M_{DR}/F^0 intervene?

5. An Assymmetry between finite and infinite places

It is expected that for any motive M in $\mathcal{MM}_{\mathbb{Z}}$ we have

$$\mathrm{Ext}^2_{\mathcal{MM}_{\mathbb{Z}}}(\mathbb{Q}(0),M)=0\,. \tag{5.1}$$

Via the isomorphism (2.2) this conjecture is compatible with the idea that $\mathcal{Y}\backslash\{\infty\}$ is an “affine curve” and therefore its second cohomology should vanish:

$$H^2(\text{“spec}\,\mathbb{Z}\text{”},\mathcal{F}(M))=0\,.$$

As shown in [D–N] assertion (5.1) for $M = \mathbb{Q}(1)$ is related to the fact that in the Arakelov Chow group $CH^1(\mathcal{Y})_{\mathbb{Q}}$ tensored with $\mathbb{Q}$ the class of any divisor with support in finite places vanishes. On the other hand the class of a divisor whose support excludes a fixed finite prime p does not need to vanish of course. Correspondingly one can show that the vanishing of $\mathrm{Ext}^2_{\mathcal{MM}_{\mathcal{Y}_p}}(\mathbb{Q}(0), \mathbb{Q}(1))$ would lead to contradictions. Superficially this seems to contradict the idea that $\mathcal{Y}_p = \mathcal{Y} \setminus p$ should be an "affine curve" as well as $\mathcal{Y} \setminus \infty$. In fact there is no contradiction because as we explained in section 2 we expect $\mathcal{F}(\mathbb{Q}(0)) \neq \mathcal{C}_{\mathcal{Y}_p}$ for the functor

$$\mathcal{F} : \mathcal{MM}_{\mathcal{Y}_p} \longrightarrow \{\text{sheaves on } \mathcal{Y}_p\}$$

since ∞ is a singular point of $\mathcal{Y}$. As a consequence

$$\mathrm{Ext}^2_{\mathcal{MM}_{\mathcal{Y}_p}}(\mathbb{Q}(0), \mathbb{Q}(1))_{\mathbb{C}} \xrightarrow{\sim} \mathrm{Ext}^2_{\mathcal{Y}_p}(\mathcal{F}(\mathbb{Q}(0)), \mathcal{F}(\mathbb{Q}(1)))^{\Theta\sim 0} \tag{5.2}$$

may very well be non–zero whereas

$$\mathrm{Ext}^2_{\mathcal{Y}_p}(\mathcal{C}_{\mathcal{Y}_p}, \mathcal{F}(M)) = H^2(\mathcal{Y}_p, \mathcal{F}(M))$$

should be zero for every M in $\mathcal{MM}_{\mathcal{Y}_p}$.

To illustrate this consider an affine curve $X_0/\mathbb{F}_q$ with a singularity in the $\mathbb{F}_q$–rational point $\infty \in X_0$ and let $j : \operatorname{spec} K_0 \to X_0$ be the inclusion of the generic point. Assume that $X_0 \setminus \infty$ is smooth. We will calculate $\mathrm{Ext}^2_X(j_*\mathbb{Q}_l, j_*\mathcal{M})$ for certain smooth $\mathbb{Q}_l$–sheaves $\mathcal{M}_0$ over $\operatorname{spec} K_0$ and show that if the curve X_0 is suitable we have

$$\mathrm{Ext}^2_X(j_*\mathbb{Q}_l, j_*\mathbb{Q}_l(1))^{\mathrm{Fr}_q} \neq 0\,. \tag{5.3}$$

On the other hand it is of course known that

$$\mathrm{Ext}^2_X(\mathbb{Q}_l, j_*\mathcal{M}) = H^2(X, j_*\mathcal{M}) = 0$$

for all $\mathbb{Q}_l$–sheaves $\mathcal{M}_0$ on $\operatorname{spec} K_0$. Note the analogies

$$\mathcal{M}_0 \mathrel{\hat{=}} M\,,\ X \mathrel{\hat{=}} \mathcal{Y}_p\,,\ j_*\mathcal{M} \mathrel{\hat{=}} \mathcal{F}(M)\,.$$

Let $\pi : \tilde{X} \to X$ be the normalization and $\tilde{j} : \operatorname{spec} K \to \tilde{X}$ the inclusion. We find

$$\mathrm{Ext}^n_X(j_*\mathbb{Q}_l, j_*\mathcal{M}) = \mathrm{Ext}^n_{\tilde{X}}(\pi^*\pi_*\mathbb{Q}_l, \tilde{j}_*\mathcal{M}) \tag{5.4}$$

since $\tilde{j}_*\mathbb{Q}_l = \mathbb{Q}_l$ as $\tilde{X}$ is smooth. Let $\triangle$ be the skyskraper sheaf defined by the exact sequence

$$0 \longrightarrow \mathbb{Q}_l \longrightarrow \pi^*\pi_*\mathbb{Q}_l \longrightarrow \triangle \longrightarrow 0\,.$$

Clearly $\triangle = \bigoplus_{\tilde{\infty} \mid \infty} i_{\tilde{\infty}*}\triangle_{\tilde{\infty}}$ where $i_{\tilde{\infty}} : \tilde{\infty} \hookrightarrow \tilde{X}$ is the closed immersion of the point $\tilde{\infty}$ lying above ∞ and hence we obtain the exact sequence

$$H^1(\tilde{X}, \tilde{j}_*\mathcal{M}) \to \prod_{\tilde{\infty} \mid \infty} \mathrm{Ext}^2_{\tilde{X}}(i_{\tilde{\infty}*}\triangle_{\tilde{\infty}}, \tilde{j}_*\mathcal{M}) \to \mathrm{Ext}^2_X(j_*\mathbb{Q}_l, j_*\mathcal{M}) \to 0\,. \tag{5.5}$$

Now

$$\mathrm{Ext}^2_{\tilde{X}}(i_{\tilde{\infty}*}\triangle_{\tilde{\infty}}, \tilde{j}_*\mathcal{M}) = \mathrm{Ext}^2_{\tilde{\infty}}(\triangle_{\tilde{\infty}}, Ri^!_{\tilde{\infty}}\tilde{j}_*\mathcal{M}).$$

Now assume for simplicity that $\mathcal{G} = \tilde{j}_*\mathcal{M}$ is smooth e.g. $\mathcal{M} = \mathbb{Q}_l(n), \mathcal{G} = \mathbb{Q}_l(n)$. Then by purity we know that

$$R^s i^!_{\tilde{\infty}}\mathcal{G} = 0 \text{ for } s \neq 2 \quad \text{and} \quad R^2 i^!_{\tilde{\infty}}\mathcal{G} = \mathcal{G}_{\tilde{\infty}}(-1).$$

From (5.5) we therefore get the exact sequence:

$$(5.6) \qquad H^1(\tilde{X}, \mathcal{G}) \to \prod_{\tilde{\infty}\,|\,\infty} \mathrm{Hom}(\triangle_{\tilde{\infty}}, \mathcal{G}_{\tilde{\infty}}(-1)) \to \mathrm{Ext}^2_X(j_*\mathbb{Q}_l, j_*\mathcal{M}) \to 0.$$

For $\mathcal{M} = \mathbb{Q}_l(1)$ we obtain

$$(5.7) \qquad H^1(\tilde{X}, \mathbb{Q}_l(1)) \to \prod_{\tilde{\infty}\,|\,\infty} \mathrm{Hom}(\triangle_{\tilde{\infty}}, \mathbb{Q}_l) \to \mathrm{Ext}^2_X(j_*\mathbb{Q}_l, j_*\mathbb{Q}_l(1)) \to 0.$$

Define X_0 to be the affine curve over $\mathbb{F}_q$ obtained by identifying two $\mathbb{F}_q$–rational points on $\mathbb{A}^1/\mathbb{F}_q$ to one point $\infty \in X_0(\mathbb{F}_q)$. Then X_0 has a singularity in ∞ and we have $H^1(\tilde{X}, \mathbb{Q}_l(1)) = 0$ since $\tilde{X} = \mathbb{A}^1$. By construction the group

$$\prod_{\tilde{\infty}\,|\,\infty} \mathrm{Hom}(\triangle_{\tilde{\infty}}, \mathbb{Q}_l)$$

is fixed by the Frobenius Fr_q and has dimension two. Using (5.7) it follows that

$$\mathrm{Ext}^2_X(j_*\mathbb{Q}_l, j_*\mathbb{Q}_l(1))^{\mathrm{Fr}_q} \neq 0.$$

6. The sign in the functional equation of the L–function of an orthogonal motive

In the following lines we explain why the cohomological formalism would imply that the global root number of an orthogonal motive over a number field is positive. The question put to T. Saito whether he could prove this if the global root number is replaced by a product of local ones resulted in his paper [S]. The argument to come is quite standard in the function field case.

So let M be a pure motive of weight n over the number field K with coefficients in the number field E. It is called orthogonal if there is a symmetric morphism $S : M \otimes M \to E(-n)$ which induces an isomorphism $M^* \cong M(n)$. For example the Artin motive attached to a representation $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_N(E)$ ist orthogonal if and only if ρ is orthogonal. For even n and $E = \mathbb{Q}$ the motive $M = H^n(X)$ where X is a smooth projective variety over K is orthogonal by Poincaré duality and hard Lefschetz. Here if $n \leq d = \dim X$ as we may assume the pairing is given by $S(x, y) = \mathrm{Tr}(x \cup l^{d-n} \cup y)$ where l in $H^2(X)(1)$ is the class of a hyperplane section and $\mathrm{Tr} : H^{2d}(X)(d-n) \to \mathbb{Q}(-n)$ is a twist of the trace map.

Let $\hat{L}(M,s)$ be the completed L–function of the orthogonal motive M and let $\varepsilon(M,s)$ be the ε–factor in the conjectured functional equation

$$\hat{L}(M,s) = \varepsilon(M,s)\ \hat{L}(M^*,1-s) = \varepsilon(M,s)\ \hat{L}(M,n+1-s)\,.$$

Note that these functions are $E \otimes \mathbb{C} = \mathbb{C}^{\mathrm{Hom}(E,\mathbb{C})}$–valued. By an index $\sigma \in \mathrm{Hom}(E,\mathbb{C})$ we will denote the σ–th component. Let

$$w(M) = \varepsilon\left(M, \frac{n+1}{2}\right) = (w_\sigma(M))$$

be the root number of M. Writing $\varepsilon(M,s)_\sigma = w_\sigma(M) b_\sigma^{\frac{n+1}{2}-s}$ and setting $D(M,s)_\sigma = b_\sigma^{\frac{s}{2}} \hat{L}(M,s)_\sigma$ the functional equation reads

$$D(M,s)_\sigma = w_\sigma(M)\ D(M,n+1-s)_\sigma\ .$$

Hence

$$w_\sigma(M) = (-1)^{r_\sigma} \quad \text{where} \quad r_\sigma = \mathrm{ord}_{s=\frac{n+1}{2}} \hat{L}(M,s)_\sigma\ . \tag{6.1}$$

We now explain how it would follow from cohomological considerations that we actually have $w(M) = 1$. For evidence supporting $w(M) = 1$ compare Theorem 3 in T. Saito's paper [S]. Since M has coefficients in E the spaces $H^i(\mathcal{Y},\mathcal{F}(M))$ would be $E\otimes\mathbb{C}$–modules. Let $H^i(\mathcal{Y},\mathcal{F}(M))_\sigma$ be the image under the idempotent in $E\otimes\mathbb{C}$ corresponding to the embedding σ. Taking the σ–components in formula (3.8) we get:

$$\hat{L}(M,s)_\sigma = \prod_{i=0}^{2} \det\nolimits_\infty \left(\frac{1}{2\pi}(s-\Theta)\,|\,H^i(\mathcal{Y},\mathcal{F}(M))_\sigma\right)^{(-1)^{i+1}}\ .$$

Since by (3.6) the weights i.e. the real parts of the eigenvalues of Θ should be $\frac{n+i}{2}$ on $H^i(\mathcal{Y},\mathcal{F}(M))_\sigma$ it follows in particular that

$$r_\sigma = \dim_{\mathbb{C}} H^1(\mathcal{Y},\mathcal{F}(M))_\sigma^{\Theta\sim\frac{n+1}{2}}\ . \tag{6.2}$$

The pairing [,] defined as the composition

$$\begin{array}{ccc}
H^1(\mathcal{Y},\mathcal{F}(M))_\sigma \times H^1(\mathcal{Y},\mathcal{F}(M))_\sigma & \xrightarrow{\cup} & H^2(\mathcal{Y},\mathcal{F}(M)\otimes\mathcal{F}(M))_\sigma \\
 & & \downarrow \\
 & & H^2(\mathcal{Y},\mathcal{F}(M\otimes M))_\sigma \\
 & & \downarrow S \\
 & & H^2(\mathcal{Y},\mathcal{F}(E(-n)))_\sigma \\
 & & \| \wr\ \mathrm{Tr} \\
 & & \mathbb{C}(-n-1)
\end{array}$$

will be perfect in the sense explained in [D1] (7.19) and alternating. Since the endomorphism Θ satisfies the Leibnitz rule with respect to $\cup$ we find

$[\Theta h, h'] + [h, \Theta h'] = (n+1)[h, h']$ and thus $\Theta - \frac{n+1}{2}\mathrm{id}$ will be skew adjoint with respect to $[,]$. Hence the induced alternating pairing

$$H^1(\mathcal{Y}, \mathcal{F}(M))_\sigma^{\Theta\sim\frac{n+1}{2}} \times H^1(\mathcal{Y}, \mathcal{F}(M))_\sigma^{\Theta\sim\frac{n+1}{2}} \longrightarrow \mathbb{C}(-n-1)^{\Theta\sim n+1} = \mathbb{C}$$

will be perfect as well, so that by (6.2) the dimension r_σ must be even. It follows from (6.1) that $w_\sigma(M) = 1$ for all σ and hence that $w(M) = 1$.

7. $\mathbf{Ext}^\nu(\mathbb{C}(\mathbf{0}), \mathbb{C}(\mathbf{s}))$

Looking at the isomorphism (2.2) one may wonder whether it is possible to define an abelian category $\mathcal{N}\mathcal{M}_{\mathcal{O}_k}$ of "motives" over $\operatorname{spec} \mathcal{O}_k$ with coefficients in $\mathbb{C}$ and a functor
$\mathcal{M}\mathcal{M}_{\mathcal{O}_k} \otimes \mathbb{C} \to \mathcal{N}\mathcal{M}_{\mathcal{O}_k}$ with the following properties:

a) For every $s \in \mathbb{C}$ there is a Tate object $\mathbb{C}(s)$ in $\mathcal{N}\mathcal{M}_{\mathcal{O}_k}$. The Tate object $\mathbb{Q}(n)$ in $\mathcal{M}\mathcal{M}_{\mathcal{O}_k}$ is mapped to $\mathbb{C}(n)$ for $n \in \mathbb{Z}$.

b) There are isomorphisms for every M in $\mathcal{M}\mathcal{M}_{\mathcal{O}_k}$ where $M(s) = M \otimes \mathbb{C}(s)$:

$$\operatorname{Ext}^\nu_{\mathcal{N}\mathcal{M}_{\mathcal{O}_k}}(\mathbb{C}(0), M(s)) \xrightarrow{\sim} H^\nu(\text{“}\operatorname{spec} \mathcal{O}_k\text{”}, \mathcal{F}(M)(s))^{\Theta\sim 0}.$$

Using (3.1) it would follow that a zero ρ of $L(M,s)$ gave rise to a non–trivial extension in $\operatorname{Ext}^1_{\mathcal{N}\mathcal{M}_{\mathcal{O}_k}}(\mathbb{C}(\rho), M^*(1))$ and vice versa if $\rho \notin \mathbb{Z}$ by (3.2). In particular for $0 < \operatorname{Re}\lambda < 1$ the group $\operatorname{Ext}^1_{\mathcal{N}\mathcal{M}_{\mathcal{O}_k}}(\mathbb{C}(0), \mathbb{C}(\lambda))$ would be non–zero if and only if λ were a non–trivial zero of the Dedekind zeta function $\zeta_k(s)$ and its dimension would equal $\operatorname{ord}_{s=\lambda}\zeta_k(s)$.

An interesting special case is the following. Our formalism implies that

$$V_k = H^1(\mathcal{Y}, \mathcal{F}(\mathbb{Q}(0)))^{\Theta\sim\frac{1}{2}}$$

is a finite dimensional $\mathbb{C}$–vector space equipped with a symplectic pairing $\cup : V_k \times V_k \to \mathbb{C}$ coming from the Poincaré duality

$$\cup : H^1(\mathcal{Y}, \mathcal{F}(\mathbb{Q}(0)))^{\Theta\sim\frac{1}{2}} \times H^1(\mathcal{Y}, \mathcal{F}(\mathbb{Q}(0)))^{\Theta\sim\frac{1}{2}} \to H^2(\mathcal{Y}, \mathcal{F}(\mathbb{Q}(0)))^{\Theta\sim 1} \overset{\mathrm{Tr}}{\cong} \mathbb{C}.$$

The dimension of V_k equals the zero–order of $\zeta_k(s)$ at $s = \frac{1}{2}$.

A consideration of relative exact sequences reveals that we should have

(7.1) $$V_k = H^1(\text{“}\operatorname{spec} \mathcal{O}_k\text{”}, \mathcal{F}(\mathbb{Q}(0)))^{\Theta\sim\frac{1}{2}} = \operatorname{Ext}^1_{\mathcal{N}\mathcal{M}_{\mathcal{O}_k}}\left(\mathbb{C}(0), \mathbb{C}\left(\tfrac{1}{2}\right)\right)$$

and in fact

$$V_k = H^1(\text{“}\operatorname{spec} k\text{”}, \mathcal{F}(\mathbb{Q}(0)))^{\Theta\sim\frac{1}{2}}$$

the inductive limit over shrinking $\mathcal{Y}_S$.

Taking into account the considerations in [D–N] we get the following suggestion:

(7.2) To k there should be naturally associated a finite dimensional $\mathbb{C}$–vector space V_k together with a symplectic pairing

$$\cup : V_k \times V_k \longrightarrow CH^1(\mathcal{Y}_k)_{\mathbb{C}} \xrightarrow{\deg} \mathbb{C} \tag{7.3}$$

such that

$$\mathrm{ord}_{s=\frac{1}{2}} \zeta_k(s) = \dim V_k \,.$$

The construction of V_k should not require the consideration of $\mathcal{O}_k$ but only of k but the construction of $\cup$ should use the arithmetic of the ring of integers.

Note that according to the functional equation $\zeta_k(s)$ vanishes to even order at $s = \frac{1}{2}$ if it vanishes at all. This is compatible with the fact that V_k is even dimensional since it carries a symplectic pairing. By Armitage [A] there are Dedekind zeta functions with a zero at $s = \frac{1}{2}$. In fact an argument of Kurokawa shows that for suitable Galois extensions $k/\mathbb{Q}$ the order of vanishing of $\zeta_k(s)$ at $s = \frac{1}{2}$ gets arbitrarily large c.f. [Mi] § 5. It seems rather strange to me that nothing is known about (7.3) whereas the conceptually much more difficult Birch and Swinnerton–Dyer conjecture has been considered for a long time. A construction of V_k would give valuable hints on the category $\mathcal{NM}$ and on $\mathbb{C}\left(\frac{1}{2}\right)$ via equation (7.1).

8. Remarks on the algebra of correspondences on $\mathcal{Y}$

The correspondences $\mathcal{Y} \dashrightarrow \mathcal{Y}$ on $\mathcal{Y}$ should form a C^*–algebra (possibly after completion) which we denote by $\mathrm{Corr}\,(\mathcal{Y})$. For a related discussion of this algebra compare [P1] (5.18). The T^+–operation on $\mathcal{Y}$ would give a map

$$\sigma : T^+ \to \mathrm{Corr}\,(\mathcal{Y}) \quad \text{such that } \sigma_{t_1+t_2} = \sigma_{t_1} \circ \sigma_{t_2} \,.$$

Let $C(T^+)$ be a class of functions $\phi : T^+ \to \mathbb{C}$ for which the integral

$$\Lambda(\phi) = \int_0^\infty \phi(t) \sigma_t \, dt$$

exists in $\mathrm{Corr}\,(\mathcal{Y})$. For a suitable subclass of continous ϕ's the $\mathbb{C}$–vector space $C(T^+)$ becomes an algebra under convolution:

$$(\phi_1 * \phi_2)(t) = \int_0^t \phi_1(\tau)\phi_2(t-\tau)\, d\tau$$

and the map

$$\Lambda : C(T^+) \longrightarrow \mathrm{Corr}\,(\mathcal{Y})$$

becomes a homomorphism of algebras.

Similarly we get an action of $C(T^+)$ on sheaves and cohomologies in our site (after completion). Note that by a result of Titchmarch e.g. [Y] VI 5 the algebra $C(T^+)$

has no zero divisors and hence no idempotents except 1. Thus the operation of $C(T^+)$ induces no "exotic" decompositions of cohomology.

The Lefschetz trace formula of [D1] (7.2) was applied to correspondences which were associated to analytic functions Φ as follows: take the infinitesimal generator Θ_σ of the semigroup σ i.e. $\sigma_t = \exp(t\Theta_\sigma)$ and insert Θ_σ into Φ using the analytic functional calculus. Then apply the trace formula to $\Phi(\Theta_\sigma)$. It was supposed that on cohomologies and stalks of sheaves this corresponds to $\Phi(\Theta)$ where Θ is the infinitesimal generator of the T^+–semigroup action on cohomologies and stalks. Deligne observed that this is only possible if a correspondence is more than a cycle because otherwise Θ_σ would induce the identity on cohomology since its support is reduced to the diagonal in "$\mathcal{Y} \times \mathcal{Y}$". This phenomenon that a correspondence is more than a cycle also appears in the work of Gillet and Soulé [G–S]. As in their work one may assume that a cycle is really a sort of current.

Finally we remark that the $\frac{2\pi i}{\log q}$–periodicity of the poles of the L–factor

$$L(H^w(X), s) = \det_{\mathbb{Q}_l}(1 - q^{-s}\mathrm{Fr}_q^* \mid H^w(X, \mathbb{Q}_l))^{-1}$$

of a variety $X_0/\mathbb{F}_q$ is also visible in the structure of the cohomology $H^w(X/\mathbb{L})$ introduced in [D1] §4: Via cup product this cohomology is a module under $H^0(X/\mathbb{L}) \supseteq \mathbb{L}$ and it carries an action of $\Theta_q = \frac{2\pi i}{\log q} z \frac{d}{dz}$. Thus the non–commutative ring $\mathbb{L}[\Theta_q]$ operates. For the element $z \in \mathbb{L} = \mathbb{C}[z, z^{-1}]$ we have in $\mathbb{L}[\Theta_q]$

$$[\Theta_q, z] = \frac{2\pi i}{\log q} z \, .$$

Hence the isomorphism "multiplication by z" maps the α–eigenspace of Θ_q on $H^w(X/\mathbb{L})$ into the $\alpha + \frac{2\pi i}{\log q}$–eigenspace. Thus also the following formula of loc. cit.

$$L(H^w(X), s) = \det_\infty((s - \Theta_q) \mid H^w(X/\mathbb{L}))^{-1}$$

implies that the poles of the L–factor are $\frac{2\pi i}{\log q}$–periodic. On the other hand in the number field case the $H^0(\mathcal{Y}, \mathcal{F}(\mathbb{Q}(0))) = \mathbb{C}$–module structure of $H^1(\mathcal{Y}, \mathcal{F}(\mathbb{Q}(0)))$ gives just its $\mathbb{C}$–vector space structure and Θ is $\mathbb{C}$–linear so that nothing follows for the eigenvalues.

9. Relations with Selberg's class

In [Se] see also [Mu] A. Selberg introduced a class $\mathcal{S}$ of functions with properties generalizing those of L–functions and made a number of far reaching conjectures about them. In particular they should factor uniquely into "primitive" functions in the class and there should be a kind of orthogonality relation between different primitive functions expressed in terms of the asymptotics of certain series. It is of course too early to speculate on the relation with our cohomological formalism but I cannot resist the temptation: On the singular curve $\mathcal{Y}$ there should exist a category $\mathrm{Perv}(\mathcal{Y})$ of perverse sheaves. It would be abelian semisimple the simple objects either having support in a closed point of $\mathcal{Y}$ or being "generic" i.e. of the form $j_{!*}\mathcal{G}$ with j : "spec $\mathbb{Q}$" $\to \mathcal{Y}$ the inclusion and $\mathcal{G}$ locally constant and simple on "spec $\mathbb{Q}$". Since $\mathcal{Y}$ is a curve we should have $j_{!*}\mathcal{G} = j_*\mathcal{G}$. Let ${}_0\mathrm{Perv}^{\mathrm{gen}}(\mathcal{Y})$ be the full abelian semisimple subcategory of $\mathrm{Perv}(\mathcal{Y})$ generated by the "generic" objects of weight zero. Then any sheaf $\mathcal{F}$ in ${}_0\mathrm{Perv}^{\mathrm{gen}}(\mathcal{Y})$ is of the form $j_{!*}\mathcal{G} = j_*\mathcal{G}$ and it decomposes uniquely into a direct sum of "generic" simple objects. Then for any $\mathcal{F}$ in ${}_0\mathrm{Perv}^{\mathrm{gen}}(\mathcal{Y})$ the function

$$\begin{aligned} F(s) &= L(\mathcal{F}, s) := \prod_{i=0}^{2} \det{}_\infty \left(\frac{1}{2\pi}(s - \Theta) \mid H^i_c(\text{“spec}\,\mathbb{Z}\text{”}, \mathcal{F}) \right)^{(-1)^{i+1}} \\ &= \prod_{i=0}^{1} \det{}_\infty \left(\frac{1}{2\pi}(s + \Theta) \mid H^i(\text{“spec}\,\mathbb{Z}\text{”}, \mathcal{F}^*(1)\right)^{(-1)^{i+1}} \end{aligned}$$

should be in Selberg's class $\mathcal{S}$. The number n_F attached to F (c.f. [Mu] p.2) which is also the residue of $L(\mathcal{F}^* \otimes \mathcal{F}, s)$ at $s = 1$ is then given by:

$$n_F = \dim_{\mathbb{C}} \mathrm{Hom}_{\text{“spec}\,\mathbb{Z}\text{”}}(\mathcal{C}, \mathcal{F}^* \otimes \mathcal{F}) = \dim_{\mathbb{C}} \mathrm{End}_{\mathcal{Y}}(\mathcal{F}) \,. \tag{9.1}$$

Here we have used an obvious generalization of (3.2)–(3.5) to sheaves $\mathcal{F}$ in ${}_0\mathrm{Perv}^{\mathrm{gen}}(\mathcal{Y})$ which are not necessarily of the form $\mathcal{F}(N)$ for a motive N. The primitive functions in Selberg's class are conjecturally those with $n_F = 1$. In the sheaf theoretic picture note that

$$\begin{aligned} \mathrm{End}_{\mathcal{Y}}(\mathcal{F}) &= \mathrm{Hom}_{\text{“spec}\,\mathbb{Z}\text{”}}(\mathcal{C}, \mathcal{F}^* \otimes \mathcal{F}) \\ &= H^0(\text{“spec}\,\mathbb{Z}\text{”}, \mathcal{F}^* \otimes \mathcal{F}) \end{aligned}$$

is expected to be finite dimensional. Thus $\mathrm{End}_{\mathcal{Y}}(\mathcal{F})$ is a finite dimensional $\mathbb{C}$–algebra. It is a skew field hence isomorphic to $\mathbb{C}$ if and only if $\mathcal{F}$ is a simple object of $\mathrm{Perv}(\mathcal{Y})$. Thus the relation

$$\dim_{\mathbb{C}} \mathrm{End}_{\mathcal{Y}}(\mathcal{F}) = 1$$

characterizes simple objects $\mathcal{F}$. Granting (9.1) it follows that simple objects in ${}_0\mathrm{Perv}^{\mathrm{gen}}(\mathcal{Y})$ should correspond to primitive functions. The orthogonality of different primitive functions in Selberg's class would correspond to the vanishing of

$$\mathrm{Hom}_{\text{“spec}\,\mathbb{Z}\text{”}}(\mathcal{C}, \mathcal{F}_1^* \otimes \mathcal{F}_2) = \mathrm{Hom}_{\mathcal{Y}}(\mathcal{F}_1, \mathcal{F}_2)$$

for non–isomorphic simple sheaves $\mathcal{F}_1, \mathcal{F}_2$.

Let ${}_0\mathcal{M}_{\mathbb{Q}}$ denote the $\mathbb{Q}$–linear category of pure motives over $\mathbb{Q}$ of weight zero. The functor

$$\mathcal{F} : \; {}_0\mathcal{M}_{\mathbb{Q}} \otimes \mathbb{C} \longrightarrow {}_0\mathrm{Perv}^{\mathrm{gen}}(\mathcal{Y}) \subset \mathrm{Perv}(\mathcal{Y})$$

is fully faithful if we admit (2.2) since then:

$$\begin{aligned} \mathrm{Hom}_{{}_0\mathcal{M}_{\mathbb{Q}}}(M_1, M_2) \otimes \mathbb{C} &= \mathrm{Hom}_{\mathcal{M}\mathcal{M}_{\mathbb{Q}}}(\mathbb{Q}(0), M_1^* \otimes M_2) \otimes \mathbb{C} \\ &\overset{(2.2)}{=} H^0(\text{“spec}\,\mathbb{Q}\text{”}, \mathcal{F}(M_1^* \otimes M_2))^{\Theta\sim 0} \end{aligned}$$

$$
\begin{aligned}
&= H^0(\text{“spec}\,\mathbb{Q}\text{”}, \mathcal{F}(M_1)^* \otimes \mathcal{F}(M_2)) \quad \text{by (3.3), (3.2), (3.6)} \\
&= \mathrm{Hom}_{\text{“spec}\,\mathbb{Z}\text{”}}(\mathcal{F}(M_1), \mathcal{F}(M_2)) \\
&= \mathrm{Hom}_{\mathcal{Y}}(\mathcal{F}(M_1), \mathcal{F}(M_2))\,.
\end{aligned}
$$

In particular $\mathcal{F}(M)$ is a simple object in the semisimple category $\mathrm{Perv}(\mathcal{Y})$ if and only if $\mathrm{End}\, M = \mathbb{Q}$. Finally note that what is called $\dim F = 2\alpha_F$ in Selberg's theory is the rank of the motive M if $F(s) = L(\mathcal{F}(M), s) = L(M, s)$.

Remark: From our geometric point of view it would be more natural if L–functions corresponding to the perverse sheaves $i_*\mathcal{G}_0$ were included in Selberg's class as well where i is the inclusion of a closed point of $\mathcal{Y}$. However these L–functions are comparatively well understood of course.

10. Some tentative remarks on constructions

The (intersection) cohomologies $H^i(\mathcal{Y}, \mathcal{F}(\mathbb{Q}(0)))$ of $\mathcal{Y}$ were recalled in § 1. They look like singular cohomology with coefficients in $\mathbb{C}$ of a compact oriented surface of infinite genus which will automatically have singularities. The singularities should be caused by the infinite primes of k. Note that as pointed out in [De1] § 7 the stalks $\mathcal{C}_p$ for $p \mid \infty$ have infinitely many weights whereas the stalks $\mathcal{C}_p$ for $p \nmid \infty$ are pure of weight zero. Geometrically the situation cannot be so simple however because a continuous action of T^+ on such a surface would be homotopic to the identity and hence induce the identity on cohomology. My impression is that the spaces in our site are real analytic spaces X in the usual sense (possibly singular and infinite dimensional) and $\mathcal{C}_X$ is a suitable subalgebra sheaf of its sheaf of real analytic sections. A similar conclusion was reached by N. Kurokawa for $\mathcal{Y}$. Both should be equipped with a compatible T^+–action. The space associated to a finite prime p of k i.e. to $\mathrm{spec}\,(\mathcal{O}_k/p)$ would be

$$
\mathbb{R}/\log Np \cdot \mathbb{Z} \cong S^1_p \qquad \bar{t} \longmapsto \exp\left(\frac{2\pi i}{\log Np} t\right)
$$

together with its sheaf of real–algebraic sections and the natural action of T^+ by translation. A p–valued point of $\mathcal{Y}$ should be a T^+–equivariant map $p : S^1_p \to X$ if X corresponds to $\mathcal{Y}$. The sheaves $\mathcal{F}(M)$ on $\mathcal{Y}$ would be coherent sheaves of $\mathcal{C}$–algebras on X. The stalk $\mathcal{F}_p$ of $\mathcal{F}$ would be the space of sections $\mathcal{F}_p = \Gamma(S^1_p, p^*\mathcal{F})$. For the stalk of the structure sheaf $\mathcal{C}$ at p we get $\mathcal{C}_p \cong \mathbb{L}_p$ as T^+–modules (c.f. [D1] § 3). Similarly the space associated to an infinite prime p would be $\mathbb{R}^+$ (not $\mathbb{R}$, see below) with its sheaf of real–algebraic sections and the natural action of T^+ by translation. A p–valued point of $\mathcal{Y}$ should be a T^+–equivariant map $p : \mathbb{R}^+ \to X$ and as before $\mathcal{F}_p = \Gamma(\mathbb{R}^+, p^*\mathcal{F})$. Again we have $\mathcal{C}_p \cong \mathbb{L}^+_p$ as T^+–modules (c.f. [D1] § 6). Note that in [D1] §§ 3,6 we expressed everything in terms of Θ whereas now we think in terms of the corresponding T^+–action.

In the above geometrical approach our cohomologies would just be usual sheaf–cohomology. On the other hand there exist cohomological ideas to interpret Ruelle zeta functions of dynamical systems [P2], [J]. As yet the constructions of cohomology are representation theoretic and only applicable to flows on certain symmetric spaces. Strikingly in connection with Ruelle's zeta function there also appears the regularized determinant of the infinitesimal generator of the induced flow on cohomology.

The objects of our site are somewhat analogous to base extensions to $\overline{\mathbb{F}}_q$ with Frobenius action of varieties X_0 over $\mathbb{F}_q$. So one may ask: what is the quotient $\mathcal{Y}_0$ of $\mathcal{Y}$ by T^+? We have one comment: a closed point $x_0 \in X_0$ splits into $\deg(x_0)$ points in $X = X_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$, a finite Frobenius orbit. The different points $x \mid x_0$ correspond to the embeddings of $\kappa(x_0)$ into $\overline{\mathbb{F}}_q$ over $\mathbb{F}_q$. Similarly I think that p defines a "point" of $\mathcal{Y}_0$ and that the points in $\mathcal{Y}$ lying over it form a T^+–orbit which is $\mathbb{R}/\log Np \cdot \mathbb{Z}$ for $p \not| \infty$ and $\mathbb{R}^+$ for $p \mid \infty$ compatibly with the above consideration. The idea that the points lying over ∞ should form a T^+–orbit suggested to think of $\mathbb{R}^+$ and not of $\mathbb{R}$ as the space associated to ∞ by the way.

In a precise sense $x_0 \in X_0$ "is" the Fr_q–orbit of points $x \mid x_0$ in X. The similar idea that p "is" the T^+–orbit $\mathbb{R}/\log Np \cdot \mathbb{Z}$ resp. $\mathbb{R}^+$ is also supported by the "holomorphic" Lefschetz formula of [D2] (3.22), (3.23): There at least for a finite prime p the "trace of $e^{z\Theta}$ on the stalk at p" is given by an integral over S^1 of traces in a line bundle over S^1. So p "is" S^1.

The various points $x \mid x_0$ come from conjugating an embedding of $\kappa(x_0) \hookrightarrow \overline{\mathbb{F}}_q$. The
T^+–orbit over p may come from a functional–analytic object with a natural $\mathbb{R}/\log Np \cdot \mathbb{Z}$ resp. $\mathbb{R}^+$–operation. Note that suitable von Neumann algebras of type III are such structures – disregarding inner automorphisms.

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