from garry.herrington@gmail.com , 1 November 2018

Problem with a Claimed Proof of the Riemann Hypothesis
> Eswaran 2018

Eswaran 2018 claims a proof of the Riemann Hypothesis under the title “The Final and Exhaustive Proof of the Riemann Hypothesis from First Principles”. This is a bold claim given the history of attempts to prove the Riemann Hypothesis and the absence in Eswaran 2018 of any evidence of independent expert review.

§ 1 Theorems on which the claimed proof is based

Theorem 1

If $s_1, s_2, s_3, \dots, s_n$ is a 1-D random walk with steps $s_n = \pm 1$ and a probability p that $s_n = +1$ then $E(|s_1 + s_2 + s_3 + \dots + s_n|) \underset{1}{\leq} n^{1/2}$ where E(x) is the expected value of x (Stillwell 2016, p.285).

Theorem 2

The Riemann Hypothesis is equivalent to $\lim_{n\to\infty} (\lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_n) / n^{1/2 + \varepsilon} \underset{2}{=} 0$

$\forall \varepsilon > 0$ where ε is independent of n (Borwein et al 2008, Theorem 1.2, p.6).

λ_n is the “Liouville function” and is defined as -1 if Ω(n) is odd and +1 if Ω(n) is even where

$\Omega(n) = m_1 + m_2 + m_3 + \dots + m_k$ when $n = p_1^{m_1} p_2^{m_2} p_3^{m_3} \dots p_k^{m_k}$ (Borwein et al 2008, Definition 1.1, p.6).

§ 2 A problem with the claimed proof

If p = ½ in Theorem 1 then relation(1) may be written, $-2n^{1/2} < s_1 + s_2 + s_3 + \dots + s_n < 2n^{1/2}$

and so $\lim_{n\to\infty} (s_1 + s_2 + s_3 + \dots + s_n) / n^{1/2 + \varepsilon} \underset{3}{=} 0$ $\forall \varepsilon > 0$ where ε is independent of n.

If $\lambda_1, \lambda_2, \lambda_3, \dots \lambda_n$ was a 1-D random walk with p= ½ ,as claimed in Eswaran 2018, then relations(2,3) would imply the Riemann Hypothesis. However $\lambda_1, \lambda_2, \lambda_3, \dots \lambda_n$ is not a 1-D random walk because the value of λ_n is determined by n whereas in a 1-D random walk $s_1, s_2, s_3, \dots, s_n$ the value of s_n is independent of n. Therefore the claim $\lambda_1, \lambda_2, \lambda_3, \dots \lambda_n$ is a 1-D random walk is false because λ_n does not satisfy an essential defining property of a 1-D random walk.

§ 3 Some History

An argument similar to the one in Eswaran 2018 that uses the Liouville function λ_n is made in Good and Churchhouse 1968 using the Möbius function μ(n) but those authors do not claim a proof of the Riemann Hypothesis.

Quote from Good and Churchhouse 1968, p. 857

"The aim of the present note is to suggest a "reason" for believing Riemann's hypothesis.

The Möbius function is defined by $\mu(n) = (-)^k$ if the positive integer n is the product of k different primes, μ(1) = 1, and μ(n) = 0 if n has any repeated factor.

It is known (see, for example, Titchmarsh [9, p. 315]) that a necessary and sufficient condition for the truth of the Riemann hypothesis is that $M(x) = O(x^{½ + ε})$, for all ε > 0, where $M(x) = \Sigma\ \mu(n)\ (n \leq x)$. The condition $M(x) = O(x^{½ + ε})$ would be true if the Möbius sequence {μ(n)} were a random sequence, taking the values -1, 0, and 1, with specified probabilities, those of -1 and 1 being equal.

More generally, if we first select a subsequence from {μ(n)} by striking out all the terms for which μ(n) = 0, and if this subsequence were 'equiprobably random', i.e. if the value -1 and 1 each had (conditional) probability ½ , then the condition $M(x) = O(x^{½ + ε})$ would still be true. Of course a deterministic sequence can at best be 'pseudorandom' in the usual incompletely defined sense in which the term is used, and of course all our probability arguments are put forward in a purely heuristic spirit without any claim that they are mathematical proofs."

End quote

The reference to Titchmarsh in the quote is given in the references below.

Good and Churchhouse 1968 is discussed in Davis and Hersh 1988 pp. 363 - 369 which mentions that an earlier paper, Denjoy 1931, “uses similar but less detailed probabilistic arguments”. Denjoy 1931 is one of the references in Eswaran 2018 but Good and Churchhouse 1968 is not.

References

1. Borwein, P., Choi,S., Rooney,B., and Weirathmueller,A., "The Riemann Hypothesis: A Resource for the Aficionado and Virtuoso Alike", Springer, 2008

2. Davis,P. and Hersh, R., “The Mathematical Experience”, Pelican Books 1988

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6. Stillwell, J., “Elements of Mathematics: From Euclid to Gödel”, Princeton University Press, 2016

7. Titchmarsh,E., “The Theory of the Riemann Zeta-Function”, Clarendon Press, Oxford,1951. MR 13, 741.

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