Temporal Coherence of Black Body Radiation

BY Y. KANO AND E. WOLF

Department of Physics and Astronomy, University of Rochester, Rochester, New York

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Abstract. The temporal complex coherence function of black-body radiation is calculated and is found to be expressible in terms of the generalized Riemann ζ-function. Curves are given which show the variation of the absolute value and of the argument of the temporal complex degree of coherence $\gamma(\tau)$ as functions of increasing time delay τ.

It is shown that the analytic continuation of $\gamma(\tau)$ has no zeros in the lower half of the complex τ plane. This result supports the theory proposed in the accompanying paper by Wolf about the possibility of determining certain energy spectra from measurements of the absolute value of the degree of coherence.

IN an interesting recent paper Bourret (1960) has studied the coherence properties of black-body radiation. Bourret derived expressions for the *correlation tensor* $\mathscr{E}_{ij}^{(r)}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2)$, which describe the correlations at time instants t_1 and t_2 between the electric field vectors $\mathbf{E}(\mathbf{r}, t)$ at two points $P_1(\mathbf{r}_1)$, $P_2(\mathbf{r}_2)$ in a volume filled with black-body radiation. The paper also includes curves which show the variation of the temporal coherence (dependence on $t_2 - t_1$) and also of the spatial coherence (dependence on $\mathbf{r}_2 - \mathbf{r}_1$).

The correlation functions discussed by Bourret are real functions. However, numerous researches relating to coherence properties of electromagnetic radiation carried out within the last few years (cf. Born and Wolf 1959, Ch. X) have demonstrated that a very appropriate measure of the degree of coherence (and also of the degree of polarization (Wolf 1959)), at least in the high frequency region of the electromagnetic spectrum, are not the real correlation functions, but rather the absolute values of the associated (suitably normalized) complex analytic signals (Gabor 1946, Born and Wolf 1959, §10.2; see also eq. (33) p. 502)

$$\mathscr{E}_{ij} = 2\{\mathscr{E}_{ij}^{(r)} + i\mathscr{E}_{ij}^{(i)}\} \tag{1}$$

where (if we suppress the explicit dependence on $\mathbf{r}_1$ and $\mathbf{r}_2$ and set $\tau = t_2 - t_1$)

$$\mathscr{E}_{ij}^{(i)}(\tau) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathscr{E}_{ij}^{(r)}(\tau')}{\tau' - \tau} d\tau' \tag{2}$$

is the Hilbert transform (conjugate function) of $\mathscr{E}^{(r)}$, P denoting the Cauchy principal value at $\tau' = \tau$. The analytic continuation of the complex correlation function. $\mathscr{E}_{ij}(\tau)$ may be readily shown to be regular in the lower half Π of the complex τ-plane.

In the present paper some of Bourret's results will be extended by examining the behaviour of the complex correlation tensor which describes the temporal coherence of black-body radiation. It will be shown that the diagonal terms of this tensor have no zeros on the real τ axis and that their analytic continuation has no zeros in the half plane Π. This result is of interest in connection with the theory put forward in the preceding paper (Wolf 1962) about the possibility of determining spectral profiles from measurements of the absolute value of the complex degree of coherence.

The real coherence tensor $\mathscr{E}_{ij}^{(r)}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) \equiv \mathscr{E}_{ij}^{(r)}(\mathbf{r}_2-\mathbf{r}_1, t_2-t_1)$ of black-body radiation is, as shown by Bourret (1960), given by

$$\mathscr{E}_{ij}^{(r)}(\mathbf{r}, \tau) = \int A(k)(k^2\delta_{ij}-k_ik_j)\cos kc\tau \exp(i\mathbf{k}\cdot\mathbf{r})\, d^3\mathbf{k} \tag{3}$$

where†

$$A(k) = \frac{\hbar c}{2\pi^2 k}\frac{1}{\exp(\alpha k)-1} \tag{4}$$

$$\alpha = \frac{\hbar c}{\boldsymbol{k}T} \tag{5}$$

c is the vacuum velocity of light, $\hbar = h/2\pi$, h Planck's constant, $\boldsymbol{k}$ Boltzmann's constant, T the absolute temperature, δ_{ij} the Kronecker symbol, $k = |\mathbf{k}|$ and the integration in (3) extends over the whole $\mathbf{k}$ space.

Now since the Hilbert transform of $\cos(kc\tau)$ is $-\sin(kc\tau)$, the complex coherence tensor is obtained by replacing $\cos(kc\tau)$ by $\exp(-ikc\tau)$ in (3) and multiplying the resulting expression by 2. This gives

$$\mathscr{E}_{ij}(\mathbf{r}, \tau) = 2\int A(k)(k^2\delta_{ij}-k_ik_j)\exp[i(\mathbf{k}\cdot\mathbf{r}-kc\tau)]\, d^3\mathbf{k}. \tag{6}$$

Consider now the special case of temporal coherence, i.e. the case when $\mathbf{r} = 0$. Because of symmetry, the off-diagonal elements of the temporal coherence tensor $\mathscr{E}_{ij}(0, \tau)$ are zero. The diagonal elements are equal to each other and each is readily shown to be given by the following expression, obtained from (6) after integrating over solid angle in $\mathbf{k}$ space and a trivial change in variables:

$$\begin{aligned}\mathscr{E}(\tau') &\equiv \mathscr{E}_{ii}(0, \tau) \quad \text{(no summation)}\\ &= \frac{8}{3\pi}\frac{(\boldsymbol{k}T)^4}{(\hbar c)^3}\int_0^\infty \frac{x^3\exp\{-x(1+i\tau')\}}{1-\exp(-x)}\,dx\end{aligned} \tag{7}$$

where

$$\tau' = \frac{c}{\alpha}\tau = \frac{\boldsymbol{k}T}{\hbar}\tau. \tag{8}$$

The integral on the right of (7) may readily be expressed in terms of the generalized Riemann ζ function

$$\zeta(s, a) = \sum_{n=0}^{\infty}\frac{1}{(a+n)^s} \tag{9}$$

and leads to the expression (cf. Whittaker and Watson 1940, p. 266)

$$\mathscr{E}(\tau') = \frac{16}{\pi}\frac{(\boldsymbol{k}T)^4}{(\hbar c)^3}\zeta(4, 1+i\tau'). \tag{10}$$

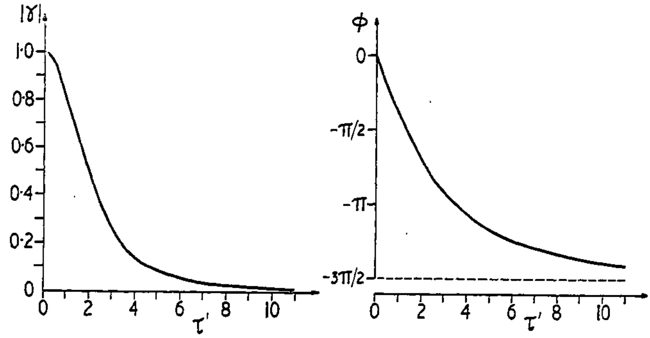
Finally on normalizing $\mathscr{E}$ in the usual way and using the fact that $\zeta(4, 1) = \pi^4/90$, one obtains the following simple expression for the (temporal) complex degree of coherence of black-body radiation:

$$\gamma(\tau') = \frac{\mathscr{E}(\tau')}{\mathscr{E}(0)} = \frac{90}{\pi^4}\zeta(4, 1+i\tau'). \tag{11}$$

† There appears to be an error in Bourret's equation (25) (arising from errors in his equations (22) and (24)) which is corrected here.

Figures (*a*) and (*b*) show the behaviour of $|\gamma|$ and of $\phi = \arg\gamma$, calculated from the formulae (11) and (9).

It is seen from figure (*a*), and may easily be proved rigorously, that γ has no real zeros. Next let us consider whether γ has any complex zeros in the lower half plane



(*a*) (*b*)

(*a*) The modulus and (*b*) the argument of the temporal complex degree of coherence $\gamma(\tau)$ of black-body radiation. For negative τ the corresponding values are obtained with the help of the formulae

$$|\gamma(-\tau)| = |\gamma(\tau)|, \quad \phi(-\tau) = -\phi(\tau).$$

Π ($\mathscr{I}\tau' < 0$). Since γ is an analytic function, regular in Π and since it has no zeros on the real τ' axis, it follows by a well-known theorem (Titchmarsh 1939) that the number Z of zeros of γ in the half plane Π is given by

$$Z = \frac{1}{2\pi}\Delta_C\phi. \tag{12}$$

Here $\Delta_C\phi$ represents the change in $\phi = \arg\gamma$ as the point τ' describes in the anti-clockwise sense the contour C, consisting of the real τ' axis (C_1 say) and the semicircle (C_2) in Π of infinite radius, centred on the origin. Now from figure (*b*) it is seen that $\phi \to \mp 3\pi/2$ as $\tau' \to \pm\infty$, so that the contribution Δ_1 to Δ_C, arising from C_1 is $+3\pi$. Moreover, by examining the asymptotic behaviour of γ one finds that, on the semicircle C_2,

$$\gamma(\tau') \sim \frac{30i}{\pi^4}\frac{1}{\tau'^3} + O\left(\frac{1}{\tau'^4}\right).$$

This implies that as the point τ' moves along C_2 from the negative end of the real τ' axis ($\arg\tau' = -\pi$) to the positive end ($\arg\tau' = 0$), the contribution Δ_2 to Δ_C arising from C_2 is -3π. Hence $\Delta_C = \Delta_1 + \Delta_2 = 0$ and it follows from (12) that $Z = 0$.† Thus the analytic continuation of γ has no zeros in the lower half of the complex τ' plane. Moreover, it is not difficult to show that $|\gamma|$ obeys the Paley–Wiener condition, and it follows (cf. Wolf 1962, equation (5)) that the argument $\phi(\tau)$ of the complex degree of coherence of black-body radiation is the 'minimal' phase function.

The result just established lends support to the theory put forward by Wolf (1962) about the possibility of determining the relative energy distribution in spectra from measurements of the absolute value of the complex degree of coherence. However, this supporting evidence cannot be relied on too heavily since the spectrum of black-body

† It is not difficult to give a more rigorous (but more lengthy) proof of this result.

radiation is rather unusual in several respects. In particular it has only one maximum and spreads over a wide frequency range. Moreover as figure (*a*) shows, the absolute value of the degree of coherence of black-body radiation decreases monotonically with the time delay τ'; this again is in contrast with the behaviour of the coherence functions of many optical spectra. Nevertheless, in view of the discussion given by Wolf (1962), our results prove conclusively that there is at least one asymmetric spectral profile, namely that of black-body radiation, which can be uniquely determined from the knowledge of the absolute value of the degree of coherence.

Finally it may be noted from (6) and (3) that

$$\mathscr{E}_{ij}(\mathbf{r}, 0) = 2\,\mathscr{E}_{ij}^{(r)}(\mathbf{r}, 0). \tag{13}$$

Hence in the special case of spatial coherence, the complex coherence tensor becomes wholly real and is then proportional to the tensor discussed by Bourret (1960). However, some of Bourret's formulae must be corrected by replacing his value for $A(k)$ by that given by our formula (4). This correction only affects the values of several proportionality factors and leaves unchanged the normalized coherence functions, the behaviour of which is displayed in the form of curves in Bourret's paper.

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