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Łukasiewicz's Logic and Prime Numbers: Introduction and Contents

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Introduction

The title of this book may seem somewhat strange, since, on the first glance, what can logics and prime numbers have in common? Nevertheless, for a certain class of finite-valued logics such commonalties do exist - and this fact has most significant consequences. But is there any link between the doctrine of logical fatalism and prime numbers?

J.Łukasiewicz's refutation of Aristotle's fatalistic argument prepared the ground for a historically first non-classical logic, namely, for the three-valued one. Its properties proved to be shocking, and its subsequent generalizations for an arbitrary finite, and further on, for an infinite cases showed that the modeling of the finite and the infinite, on the basis of Łukasiewicz's many-valued logics, yield results that justify the claim that, by the end of the twentieth century, there have taken shape and are now rapidly growing, two distinct and deep trends in the contemporary symbolic logic, namely, Łukasiewicz's finite-valued logics and Łukasiewicz's infinite-valued logic.

The book consists of four parts: (1) Łukasiewicz's finite-valued logics L_{n+1} - chapters 1-4; (2) their connection with prime numbers - chapters 5-8; (3) the issuing numeric tables; (4) an appendix on the properties of Łukasiewicz's infinite-valued logic L_∞.

In chapter 1 an elementary introduction to the classical propositional logic is given, then goes a detailed discussion of the origin and development of Łukasiewicz's three-valued logic L_3 in chapter 2, L_3 is being compared with classical logic there. Already at this stage it becomes clear that as soon as we introduce some novelties into the classical logic, there arises a pressing problem of what interpretations of logical connectives and of truth-values themselves are intuitively acceptable. This problem, in turn, leads to the question of what is logical system. More than that: in the light of subsequent results there arises the question of what is logic itself (this question is discussed in [Karpenko A.S. Logic at the border-line of millenium.

Logical Investigations 7 (2000), 7-60]. In chapter 3 the issual properties of L_{n+1} are considered, including the degree of cardinal completeness of L_{n+1} and McNaughton's criterion for definability of operations in L_{n+1}. Let us note that neither axiomatic method, nor algebraic one, nor any other semantic approaches mark divulge the uniqueness and peculiarity of Łukasiewicz's finite-valued logics L_{n+1}. The approaches of this kind we call external. The only means to penetrate into the essence of logic is to represent it as a functional system.

It was exactly this approach that allowed to discover that functional properties of L_{n+1} are highly unusual. V.K.Finn was first to note this in his brief note "On classes of functions that correspond to Łukasiewicz's M-valued logics" (1970). One corollary of his observations is that the set of functions of logic L_{n+1} is functionally precomplete if and only if n is a prime number (Ch. 5). This result (later on it was re-discovered) constitutes both the foundation of this book and the main motivation for writing it. The above result issued in developing a straightforward algorithm, which maps an arbitrary natural number to a prime one on the basis of the Euler function $\varphi(n)$. This induces a partition of the set of natural numbers into classes of equivalence. Each of those classes can be represented by a rooted tree whose root is some prime number n. That, in turn, resulted in an algorithm which maps an arbitrary prime number p to a class of equivalence whose members are natural numbers. The algorithm is based upon the properties of the invers function of Euler $\varphi^{-1}(m)$. Chapter 6 contains graphs for the first 25 prime numbers and canceled rooted trees for prime numbers from 101 (No 26) to 541 (No 100). Thus, each prime number has a structure, namely, an algebraic structure in form of *p*-Abelian groups.

Further investigations resulted in constructing such a finite-valued logic K_{n+1} that posesses the class of tautologies if and only if n is a prime number. In actual fact, the above statement constitutes a definition of prime numbers in purely logical terms. By its functional properties, logic K_{n+1} happens to coincide with L_{n+1} for the case in which n is a prime number. This provides a basis for constructing Scheffer stroke for prime numbers. One result of this is the emergence of formulas with the number of occurences of Scheffer stroke equal to 648 042 744 959. Interestingly, a combination of matrix logics for prime numbers helps discover a law of generation of classes of prime numbers. As a result, we get a partition of the set of prime numbers into equivalence classes that are induced by algebraic-logical properties of Łukasiewicz implication.

Finally, in the chapter 8 an ultimate answer is given to the question of what is Łukasiewicz's many-valued logic. Its nature is purely number-theoretical; this is why it proves possible to characterize, in terms of Łukasiewicz's logical matrices, such subsets of the set of natural numbers as prime numbers, powers of prime numbers, odd numbers, and - most arduous task! - even numbers. In the latter case, one proceeds by establishing a link with Goldbach's problem of representation of an even number by the sum of two prime numbers.

All the numerical tables have never been published before. Table 2 contains the values of the inverse function of Euler for $m \leq 5000$. This table is enough to construct graphs which correspond to the first 52 prime numbers. In the table 3 the cardinality values of rooted trees and of canceled rooted trees are listed for $p \leq 1000$. Table 4 presents the values of function $i(p)$, for $p \leq 500$, which performs partition of the set of all prime numbers to the equivalence classes. Some natural numbers are happen to be "elite", this is why of special interest is the table 1 containing the values of cardinal completeness for Łukasiewicz's logics. There have been developed special computer programs to compute these values, to construct rooted trees and to generate classes of prime numbers. Without these special programs the creation of tables wouldn't be possible. In the Appendix, some main results concerning Łukasiewicz's infinite-valued logic L_∞ are discussed; they make it very clear how deep and significant that logic is. For the only pre-table extension of L_∞, a precise model in the form of a factor-semantic is given. A separate section is dedicated to the properties of Łukasiewicz's implication; a new axiomatization of the implicative fragment of L_∞ and proof of its independence are given.

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