Sum of the reciprocals of famous series: mathematical connections with some sectors of theoretical physics and string theory

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Abstract

\textit{In this paper it has been calculated the sums of the reciprocals of famous series. The sum of the reciprocals gives fundamental information on these series. The higher this sum and larger numbers there are in series and vice versa. Furthermore we understand also what is the growth factor of the series and that there is a clear link between the sums of the reciprocal and the "intrinsic nature" of the series. We have described also some mathematical connections with some sectors of theoretical physics and string theory}
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1. **KEMPNER SERIES**

The Kempner series "9" is a modification of the harmonic series, formed by omitting all those denominators containing the figure equal to "9":

\[ S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{20} + \ldots = 22,920,676,619,264,1 \]

This number divided by 34, which is a Fibonacci number, gives as value 0.674137 value very near to the spin of the final black hole produced by the collision of two black holes and calculated from the observations of gravitational waves.

The number of elements \( N(x) \) less than or equal to \( x \) is given by the following formula:

\[ N(x) \leq 9^{\log(x)} = 9^{\frac{\ln x}{\ln 9}} \]

\[ N(100) = 81 \]
\[ N(1000) = 729 \]

The \( n^{th} \) element is given by the inverse formula that's the following:

\[ x = 10^{\log(N)} = 10^{\frac{\ln N}{\ln 9}} \]

\[ x(10000) = 15,553 \]
For the precision according to the digit which is omitted we have the following reciprocal sums:

TAB. 1

<table>
<thead>
<tr>
<th>d</th>
<th>sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>23.10344</td>
</tr>
<tr>
<td>1</td>
<td>16.17696</td>
</tr>
<tr>
<td>2</td>
<td>19.25735</td>
</tr>
<tr>
<td>3</td>
<td>20.56987</td>
</tr>
<tr>
<td>4</td>
<td>21.32746</td>
</tr>
<tr>
<td>5</td>
<td>21.83460</td>
</tr>
<tr>
<td>6</td>
<td>22.20559</td>
</tr>
<tr>
<td>7</td>
<td>22.49347</td>
</tr>
<tr>
<td>8</td>
<td>22.72636</td>
</tr>
<tr>
<td>9</td>
<td>22.92067</td>
</tr>
</tbody>
</table>

In general when we exclude strings of length n by the reciprocals, the sum is approximately given by the following formula:
\[ S = 10^n \ln 10 \]

In fact, for a single digit the sum \( S \) is given by:

\[ S = 10 \ln 10 = 23.0258509299 \]

which corresponds approximately to the values of TAB. 1
2. SEXY PRIME NUMBERS

Two prime numbers are called sexy when their difference is equal to six, or form pairs like:

\[(p, p+6)\]

where \(p\) is the lower prime.

The sum of the reciprocals is given by:

\[
S = \frac{1}{5} + \frac{1}{11} + \frac{1}{7} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \frac{1}{23} + \frac{1}{29} + \frac{1}{31} + \frac{1}{37} + \frac{1}{41} + \frac{1}{43} + \frac{1}{47} + \frac{1}{53} + \frac{1}{59} + \frac{1}{61} + \frac{1}{67} + \frac{1}{73} + \frac{1}{79} + \frac{1}{83} + \frac{1}{89} + \frac{1}{97} + \frac{1}{103} + \ldots = 1,77337685333434 \text{ (for the first 150 pairs of sexy prime)}
\]

This number divided 2, which is a Fibonacci’s number, give the value 0.886685 very near to the size of the proton.

*It is estimated that the value should be higher than 2*

\[S > 2\]

The number of elements \(N(x)\) less than or equal to \(x\) is given by the following formula:

\[
N(x) \leq 4 \times C_2 \frac{x}{(\ln x)^2} = 2,6406 \frac{x}{(\ln x)^2}
\]

Where \(C_2\) is the **constant of twin primes** = 0,6601611815

\[
N(100) = 15 \text{ (calculated } \approx 12,45) \\
N(1000) = 74 \text{ (calculated } \approx 55,34)
\]

The \(n\)th element is given by the inverse formula that’s the following:
\[ x \approx 0.66016 \ln N \cdot (\ln N)^2 \]

\[ x(10000) = 554893 \text{ (calculated } \approx 560016.169) \]
3. TWIN PRIME NUMBERS

Two prime numbers are said twin when their difference is equal to two, or form pairs like:

\((p, p+2)\)

where \(p\) is the lower prime.

The sum of the reciprocals is given by:

\[
S = \frac{1}{3} + \frac{1}{5} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \frac{1}{31} + \frac{1}{41} + \frac{1}{43} + \frac{1}{59} + \frac{1}{61} + \frac{1}{71} + \frac{1}{73} + \frac{1}{101} + \frac{1}{103} + \ldots = 1.902160583104
\]

This number divided by 3, which is also a Fibonacci’s number, gives as value 0.634 also this value very near to the spin of the final black hole produced by the collision of two black holes and calculated from the observations of gravitational waves.

The number of elements \(N(x)\) less than or equal to \(x\) is given by the following formula:

\[
N(x) \leq 2 \times C_2 \left(\frac{x}{\ln x}\right)^2 = 1.3203 \left(\frac{x}{\ln x}\right)^2
\]

where \(C_2\) is the constant of twin primes = 0.6601611815

\[
N(100) = 8 \quad \text{(calculated } \approx 6.225) \\
N(1000) = 35 \quad \text{(calculated } \approx 27.67) 
\]

The \(n\)th element is given by the inverse formula that’s the following:
\[ x \approx 1.3203 \times (\ln(N))^2 \]

\[ x(10000) = 1260989 \text{ (calculated } \approx 1120015.37) \]
4. **Cousin Prime Numbers**

Two prime numbers are called cousin when their difference is equal to four, that form pairs like:

\[(p, p+4)\]

where \(p\) is the lower prime.

The sum of the reciprocals is given by:

\[S=\frac{1}{3}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}+\frac{1}{23}+\frac{1}{37}+\frac{1}{41}+\frac{1}{43}+\frac{1}{47}+\frac{1}{67}+\frac{1}{71}+\frac{1}{79}+\frac{1}{83}+\frac{1}{97}+\frac{1}{101} + \ldots = 1.67323537619\]

This number divided by 2 gives as value 0.8366175 very near to the dimension of the proton. This number is also near to the aurea ratio 1.618…

The number of elements \(N(x)\) less than or equal to \(x\) is given by the following formula:

\[N(x) \leq 2*C_2 \left(\frac{x}{\ln x}\right)^2 = 1.3203 \left(\frac{x}{\ln x}\right)^2\]

Where \(C_2\) is the **constant of twin primes** = 0.6601611815

\[N(100) = 8 \quad (\text{calculated } \approx 6.225)\]
\[N(1000) = 41 \quad (\text{calculated } \approx 27.67)\]
The $n$th element is given by the inverse formula that’s the following:

$$x \approx 1,3203 \, N(\ln N)^2$$

$$x(10000) = 1266487 \text{ (calculated } \approx 1120015,37)$$
5. **PALINDROMIC NUMBERS**

A **palindromic number** is a number that remains the same when its digits are reversed.

An example of a palindrome number can be:

12345654321

it can be noted in fact that it is symmetrical with respect to its center:

12345 6 54321

and then it applies the definition.

The sum of the reciprocals is given by:

\[ S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{11} + \frac{1}{12} + \frac{1}{22} + \frac{1}{33} + \frac{1}{44} + \frac{1}{55} + \frac{1}{66} + \frac{1}{77} + \frac{1}{88} + \frac{1}{99} + \frac{1}{101} + \frac{1}{111} + \ldots = 3,37028325949737 \]

This number divided by 4 gives as value 0.84257 very near to the dimension of the proton. Furthermore, this number is also very near to the \( \pi \), fundamental in string theory.

The number of elements \( N(x) \) less than or equal to \( x \) is given by the following formula:

\[
N(x) \leq 2 \left( \sqrt{x} - 1 \right) \quad \text{for even exponent of } 10^x
\]

\[
N(x) \leq 11 \sqrt[10]{\frac{x}{10}} - 2 \quad \text{for odd exponent of } 10^x
\]

\[
N(100) = 18
\]
\[
N(1000) = 108
\]
The $n$th element is given by the inverse formula that’s the following:

$$x \approx \left( \frac{N + 2}{2} \right)^2$$

for even exponent of $10^n$

$$x \approx 10 \left( \frac{N + 2}{11} \right)^2$$

for odd exponent of $10^n$

$x(10000) = 8999998$ (calculated $\approx 8267768.92$)

In this case, the second formula is used because the value of $10000^{th}$ element is the closest to $10^7$
6. PALINDROMIC PRIME NUMBERS

A palindromic prime is a prime number that is also a palindromic number, which remains unchanged reading it from right to left.

Whereas the divisibility test for 11, it can be easily deduce that all palindromic numbers with an even number of digits are divisible by 11 and, therefore, are not primes; only those with a number of odd digits are palindromic prime.

The sum of the reciprocals is given by:

\[ S = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{101} + \frac{1}{131} + \frac{1}{151} + \frac{1}{181} + \frac{1}{191} + \frac{1}{313} + \frac{1}{353} + \frac{1}{373} + \frac{1}{383} + \frac{1}{727} + \frac{1}{757} + \frac{1}{787} + \frac{1}{797} + \frac{1}{919} + \frac{1}{929} + \frac{1}{10301} + \frac{1}{10501} + \frac{1}{10601} + \frac{1}{11311} + \frac{1}{11411} + \frac{1}{12421} + \ldots = 1,32398214680585 \]

This number divided by 2, which is a Fibonacci’s number, gives as value 0,6615 also this value very near to the spin of the final black hole produced by the collision of two black holes and calculated from the observations of gravitational waves.

The number of elements \( N(x) \) less than or equal to \( x \) is given by the following formula:

\[
N(x) \leq P(x) \left( \frac{\ln \ln x}{\ln x} \right) \quad \text{with } P(x) \text{ number of palindromic numbers}
\]

\[ N(100) = 5 \quad (\text{calculated } \approx 4,99) \]
\[ N(1000) = 20 \quad (\text{calculated } \approx 36,82) \]

The 10000\text{th} element is the following

\[ x(10000) = 13649694631 \]
7. **PERFECT POWER WITH DUPLICATIONS**

A **perfect power** is a positive integer that can be expressed as a power of another positive integer.

More formally, $n$ it is a perfect power if there are natural numbers $m > 1$ and $k > 1$ such that $n = m^k$.

In the case in which $k=2$ we will have the perfect squares, in the case of $k=3$ we will have the perfect cubes.

The number 1 in general is not considered (because $1^k = 1$ for any $k$).

The sum of the reciprocals with duplicates is given by:

$$S = \frac{1}{4} + \frac{1}{8} + \frac{1}{9} + \frac{1}{16} + \frac{1}{16} + \frac{1}{25} + \frac{1}{27} + \frac{1}{32} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{81} + \frac{1}{81} + \frac{1}{100} + \ldots = 1$$

The number of elements $N(x)$ less than or equal to $x$ is given by the following formula:

$$N(x) \leq 1.01 \sqrt{x}$$

$N(100) = 16$ (calculated $\approx 10.1$)

$N(1000) = 49$ (calculated $\approx 31.94$)

The $n$th element is given by the inverse formula that’s the following:

$$x \approx \left(\frac{N}{1.01}\right)^2$$

$x(9999) = 87403801$ (calculated $\approx 98010000$)
8. PERFECT POWER P-1 WITHOUT DUPLICATIONS

Euler and Goldbach have proven that the sum S of the reciprocals of $\frac{1}{p-1}$ excluding the value 1 and no duplication is given by:

$$S = 1/3 + 1/7 + 1/8 + 1/15 + 1/24 + 1/26 + 1/31 + 1/35 + 1/48 + 1/63 + 1/80 + 1/99 + ... = 1$$

The number of elements $N(x)$ less than or equal to x is given by the following formula:

$$N(x) \leq \sqrt{x}$$

$N(100) = 12$ (calculated $\approx 10$)
$N(1000) = 40$ (calculated $\approx 31.62$)

The nth element is given by the inverse formula that’s the following:

$$x \approx N^2$$

$$x(9999) = 90706575$$ (calculated $\approx 99980001$)
9. PERFECT POWER WITHOUT DUPLICATIONS

The sum $S$ of the reciprocals of the powers perfect without duplicates is given by:

$$S = \frac{1}{4} + \frac{1}{8} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{27} + \frac{1}{32} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \frac{1}{81} + \frac{1}{100} + \ldots = 0,874464368$$

This value is very near to the value of the dimension of the proton

The number of elements $N(x)$ less than or equal to $x$ is given by the following formula:

$$N(x) \leq \sqrt{x}$$

$N(100) = 12$ (calculated $\approx 10$)
$N(1000) = 40$ (calculated $\approx 31,62$)

The $n$th element is given by the inverse formula that’s the following:

$$x \approx N^2$$

$x(9999) = 90706576$ (calculated $\approx 99980001$)
10. PERFECT SQUARES (BASEL PROBLEM)

The Basel problem asks to discover the formula to which tends the sum of the inverses of all the squares of the natural numbers, or the precise sum of the infinite series:

\[ S = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{6} \]

Euler proved that the exact sum is \( \frac{\pi^2}{6} \) and announced this discovery in 1735.

\[ S = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \frac{1}{81} + \frac{1}{100} + \ldots = 1,644934066848 = \frac{\pi^2}{6} \]

The value is also equal to the Riemann zeta function \( \zeta(2) \).

This number divided by 2 gives as value 0,8224 very near to the dimension of the proton and is also near to the value of the aurea ratio 1,618…

The number of elements \( N(x) \) less than or equal to \( x \) is given by the following formula:

\[ N(x) \leq \sqrt{x} \]

\[ N(100) = 10 \quad (\text{calculated} = 10) \]
\[ N(1000) = 31 \quad (\text{calculated} \approx 31,62) \]

The nth element is given by the inverse formula that’s the following:

\[ x \approx N^2 \]

\[ x(10000) = 100000000 \quad (\text{calculated} = 100000000) \]
11. PERFECT CUBES (APERY'S CONSTANT)

The sum of the reciprocals of perfect cubes is given by:

\[ S = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \frac{1}{216} + \frac{1}{343} + \frac{1}{512} + \frac{1}{729} + \frac{1}{1000} + \ldots = 1.20205690315959 \]

This number divided by the square root of 2, gives as value about 0.85 value very near to the dimension of the proton.

The value is also equal to the Riemann zeta function \( \zeta(3) \) and is also defined as Apery's constant, which in 1978 has proven that it is an irrational number but it is still not known whether this constant is transcendental.

The number of elements \( N(x) \) less than or equal to \( x \) is given by the following formula:

\[ N(x) \leq \sqrt[3]{x} \]

\[ N(100) = 4 \quad (\text{calculated } \approx 4.64) \]
\[ N(1000) = 10 \quad (\text{calculated } = 10) \]

The nth element is given by the inverse formula that’s the following:

\[ x \approx N^3 \]

\[ x(10000) = 1000000000000 \quad (\text{calculated } = 1000000000000) \]
12. FIBONACCI NUMBERS

The Fibonacci sequence, denoted by $F_n$, is a sequence of positive integers in which each number is the sum of the previous two and the first two terms of the sequence are by definition $F_1=1$ and $F_2=1$. This sequence thus has a recursive definition according to the following rule:

$$F_1=1$$
$$F_2=1$$
$$F_n = F_{n-1} + F_{n-1} \quad \text{(for every } n>2)$$

The sum of the reciprocals is given by:

$$S = 1 + 1/2 + 1/3 + 1/5 + 1/8 + 1/13 + 1/21 + 1/34 + 1/55 + 1/89 + \ldots = 3,359885666243$$

This number divided by 4 gives as value 0.83997, about the dimension of the proton. Furthermore, this number is also near to the value of $\pi$.

The number of elements $N(x)$ less than or equal to $x$ is given by the following formula:

$$N(x) \leq \frac{1}{\ln \varphi} \ln(x\sqrt{5}) = 2.078 \ln(x\sqrt{5})$$

where $\varphi = \frac{1 + \sqrt{5}}{2} = 1.618033988749$ (golden ratio)

$N(100) = 11$ (calculated $\approx 11.24$)
$N(1000) = 16$ (calculated $\approx 16.02$)

The $n$th element is given by the inverse formula that’s the following:

$$x \approx \frac{1}{\sqrt{5}} e^{N/2.078}$$

$x(10000) = 4*10^{2089}$ (calculated $= 4*10^{2089}$)
13. POLYGONAL NUMBERS

A polygonal number is a number that can be figuratively willing to depict a regular polygon.

The general formula for the sum of the reciprocals is given by:

\[
S= \frac{2 \ln 2 + \psi\left(\frac{1}{k-2}\right) + \psi\left(\frac{k}{2k-2}\right) + 2\gamma}{k-2}
\]

where

\( \psi \) is the digamma function that is the special function defined as logarithmic derivative of the gamma function

\( \lambda = 0.57721 56649 \) (Eulero-Mascheroni constant)

For example the sum of the reciprocals of the heptagonal numbers is given by:

\[
S=1+1/7+1/18+1/34+1/55+1/81+1/112+1/148+1/189+1/235+1/286+1/342+1/403+1/469+1/540+1/616+1/697+1/783+1/874+1/970+1/1071+1/1177 +\ldots = 1.30476318377875
\]

This number divided by 2, which is a Fibonacci’s number, gives as value 0.65235 also this value very near to the spin of the final black hole produced by the collision of two blacks holes and calculated from the observations of gravitational waves.

The number of elements \( N(x) \) less than or equal to \( x \) is given by the following formula:

\[
N(x) \leq \frac{\sqrt{8(k-2)x + (k-4)^2} + k - 4}{2(k-2)}
\]

For the heptagonal numbers we have:
\begin{equation}
\sqrt{40x + 9} + 3
\end{equation}

\[
N(x) \leq \frac{10}{2}
\]

\[
N(100) = 6 \quad \text{(calculated } \approx 6.63) \\
N(1000) = 20 \quad \text{(calculated } \approx 20.3)
\]

The nth element is given by the inverse formula that’s the following:

\[
x = \frac{N^2(k - 2) - N(k - 4)}{2}
\]

The nth element for the \textbf{heptagonal numbers} is given by the inverse formula that’s the following:

\[
x = \frac{5N^2 - 3N}{2}
\]

\[
x(10000) = 249985000 \quad \text{(calculated } = 249985000 \text{ is obviously the same value)}
\]
14. **POWER OF 2**

A **power of 2** is any integer number power of 2, or that it can get by multiplying 2 by itself a certain number of times. A power of two is also 1, because $2^0=1$.

The sum of the reciprocals is given by:

$$S=1+1/2+1/4+1/8+1/16+1/32+1/64+1/128+1/256+1/512+1/1024 +... = 2$$

The number of elements $N(x)$ less than or equal to $x$ is given by the following formula:

$$N(x) \leq \frac{\ln x}{\ln 2} + 1 = 1,442695 \ln(x) + 1$$

$N(100) = 7$ (calculated $\approx 7,64$)
$N(1000) = 10$ (calculated $\approx 10,96$)

The nth element is given by the inverse formula that’s the following:

$$x \approx e^{\frac{N-1}{1,442695}}$$

$x(10000) = 9,975315584403*10^{3009}$ (calculated $= 9,975315584403*10^{3009}$ is obviously the same value)
15. FACTORIAL

It defines \textit{factorial} of a natural number \( n \), denoted by \( n! \), the product of the positive integers smaller than or equal to this number. In formula:

\[ n! = 1 \times 2 \times 3 \times \ldots \times (n-1) \times n \]

The value of \( 0! = 1 \), according to the convention for an empty product.

The sum of the reciprocals is given by:

\[ S = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40320} + \frac{1}{362880} + \frac{1}{3628800} + \ldots \approx e \]

\[ e = 2.718281828459 \]

This number divided by \( \pi = 3.14 \) gives as value 0.8656 very near to the dimension of the proton.

The sum is equal to the transcendental and irrational number \( \text{neperian } "e" \)

\[ N(100) = 5 \]
\[ N(1000) = 7 \]

To find the \( n \)th element we are using Stirling's approximation:

\[ x! \approx \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \]
x(10000) \approx 10^{35659} \text{ (calculated } \approx 10^{35659})
16. PRIMORIAL

For \( n \geq 2 \), the **primorial** of \( n \), denoted by \( n# \), is the product of all prime numbers less than or equal to \( n \). For example, 210 is a primorial, being the product of the first four primes \((2 \times 3 \times 5 \times 7)\).

The sum of the reciprocals is given by:

\[
S = \frac{1}{2} + \frac{1}{6} + \frac{1}{30} + \frac{1}{210} + \frac{1}{2310} + \frac{1}{30030} + \frac{1}{510510} + \frac{1}{9699690} + \frac{1}{223092870} + \frac{1}{6469693230} + \ldots = 0,7052301717918
\]

This value is very near to the spin of the final black hole produced by the collision of two black holes and calculated from the observations of gravitational waves.

The number of elements \( N(x) \) less than or equal to \( x \) is given by the following formula:

\[
N(x) \leq \frac{\ln x}{\ln \ln x} + 1
\]

\[
N(100) = 3 \quad \text{(calculated } \approx 3,01) \\
N(1000) = 4 \quad \text{(calculated } \approx 3,57)
\]

To find the \( nth \) element we use the following approximation:

\[
x# \approx n^{1,01n}
\]

\[
10000# \approx \text{calculated } \approx 10000^{10100} = 10^{40400}
\]
17. FIBONACCI PRIME NUMBERS

A Fibonacci prime is a Fibonacci number that is prime, a type of integer sequence prime.

Since $F_{nm}$ is divisible by $F_n$ and $F_m$, if a number is prime $F_k$, $k$ is also prime, except for $F_4=3$.

But the converse is not true. For example 19 is prime, while $F_{19} = 113 \times 37 = 4181$ is not prime.

The sum of the first 14 reciprocals is given by:

$$S = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{13} + \frac{1}{89} + \frac{1}{233} + \frac{1}{1597} + \frac{1}{28657} + \frac{1}{514229} + \frac{1}{433494437} + \frac{1}{12971215073} + \frac{1}{199194853094755497} + \frac{1}{1066340417491710595814572169} + \frac{1}{19134702400093278081449423917} + \ldots = 1,126447227672853338601666004139$$

It is estimated that the value tends to

1,126447227672853338601666004139

This number divided by 1,618, i.e. the aurea ratio, gives as value 0,69619 also this value very near to the spin of the final black hole produced by the collision of two black holes and calculated from the observations of gravitational waves.

The number of elements $N(x)$ less than or equal to $x$ is given by the following formula:
N(x) ≤ 2φ ln (ln (x \sqrt{5})) = 3.236 ln (ln (x^{\sqrt{5}}))

\[
φ = \frac{1 + \sqrt{5}}{2} = 1.618033988749 \text{ (golden ratio)}
\]

N(100) = 5 \text{ (calculated ≈ 5.46)}

N(1000) = 6 \text{ (calculated ≈ 6.61)}

The nth element is given by the inverse formula that’s the following:

\[
x ≈ \frac{1}{\sqrt{5}} e^{\frac{2}{5}}
\]

x(10000) ≈ calculated ≈ 10^{34100}
18. FERMAT NUMBERS

A Fermat number, named after the French mathematician Pierre de Fermat, is an integer number expressed as:

$$F_n = 2^{2^n} + 1$$

With \( n \) non-negative integer.

They are all odd numbers coprime to each other.

The sum of the reciprocals is given by:

$$S = \frac{1}{3} + \frac{1}{5} + \frac{1}{17} + \frac{1}{257} + \frac{1}{65537} + \frac{1}{4294967297} + \cdots = 0.596063172117821$$

We note that \( \sqrt{2} - 0.5960631 \ldots \) gives 0.82421 value very near to the proton dimension.

The number of elements \( N(x) \) less than or equal to \( x \) is given by the following formula:

$$N(x) \leq \sqrt{2} \ln (\ln x) + 1$$

\( N(100) = 3 \) (calculated \( \approx 3.159 \))
\( N(1000) = 4 \) (calculated \( \approx 3.73 \))

The \( n \)th element is given by the inverse formula that’s the following:

$$x \approx \frac{1}{\sqrt{2}} e^{\sqrt{2} \pi}$$
x(10000) ≈ calculated ≈ 10^{10^{4341}}
19. EXPONENTIAL FACTORIAL

An exponential factorial is a positive integer n that is raised to the power of n-1, which in turn is raised to the power of n-2 and so on:

\[(n-1)^{(n-2)^{\cdots}}\]  

n

The exponential factor can also be expressed as a recursive relationship:

\[a_1=1, \ a_n=n^{a_{n-1}}\]

\[a_1=1\]
\[a_2=2^1\]
\[a_3=3^2\]
\[a_4=4^{3^2}=4^9\]

The sum of the reciprocals is given by:

\[S=1+1/2+1/9+1/262144+\ldots=1.611114925808\]

It is a transcendental number and it is very near to the number 1,618... i.e. to the aurea ratio.

\[N(100) = 4\]
\[N(1000) = 4\]

\[x(10000) \approx 10000^{9999^{9998}}\] (calculated \(\approx 10^{10^{109}}\))

It’s the highest value that can be found in this paper.
20. FIBONORIAL NUMBERS

The **Fibonorial number** \( n!_F \), also called as Fibonacci factorial, where \( n \) is a non-negative integer, is defined as the product of the first \( n \) Fibonacci numbers:

\[
 n!_F = \Pi F_i, \quad n \geq 1 \quad \text{e} \quad 0!_F = 1
\]

where \( F_i \) is the \( i \)-th Fibonacci number.

The sum of the first 18 reciprocals is given by:

\[
 S = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{30} + \frac{1}{240} + \frac{1}{3120} + \frac{1}{65520} + \frac{1}{2227680} + \frac{1}{742901120} + \frac{1}{12252248740631184912217600} + \frac{1}{13262248740631184912217600} + \frac{1}{34269650745790981813170278400} + \ldots = 2,70450289915406
\]

It is estimated that the value tends to

\[
 2,704502899154067487197548966182
\]

This number divided for \( \pi = 3,14\ldots \) gives as value 0,8613 that is very near to the proton dimension

The \( n \)-th element is given by the following formula:

\[
 x \approx 1,2267420107 \left( \frac{n(n+1)}{\phi^2 5} \right)
\]

where \( \phi = \frac{1 + \sqrt{5}}{2} = 1,618033988749 \) (golden ratio)
$x(10000) \approx \text{calculated} \approx 10^{10446932}$
21. FIBONACCI SEQUENCE OF COWS - SUPERGOLDEN RATIO

It is associated with a problem (similar to that of rabbits) concerning the population of a herd of cattle.

Unlike couple of bunnies (which became adult and self-sustaining after the passage of a single month), different in this case the growth process has an intermediate stage: the pairs of pups before they turn into adult couples but not yet fertile, and then in fertile couples, capable of reproduction.

The succession on the population of bovine animals will be:

\[ 1 \; 1 \; 1 \; 2 \; 3 \; 4 \; 6 \; 9 \; 13 \; 19 \; 28 \; 41 \; 60 \]

\[ a(0) = a(1) = a(2) = 1; \text{ thereafter } a(n) = a(n-1) + a(n-3). \]

In this case, the generation blowing a value.

For example, 41 = 28 + 13, while 60 = 41 + 19.

If, as in the case of the Fibonacci sequence, we execute the relationship between each term of the sequence and the antecedent, then this ratio, taken to the limit, tends to a certain value:

\[ \Psi = 1.46557123187676802665... \]

\[ \frac{a(n+1)}{a(n)} \text{ tends to } x = 1.46557123187676802665... \text{ when } n \to \infty. \]

This is the real solution of the equation

\[ x^3 - x^2 - 1 = 0. \]

This value indicated by the Greek letter psi (\( \Psi \)) is the so-called "supergolden ratio".
The sum of the first 44 reciprocals is given by:

\[ S = 1 + 1 + 1/2 + 1/3 + 1/4 + 1/6 + 1/9 + 1/13 + 1/19 + 1/28 + 1/41 + 1/60 + 1/88 + 1/129 + 1/189 + 1/277 + 1/406 + 1/595 + 1/872 + 1/1278 + 1/1873 + 1/2745 + 1/4023 + 1/5896 + 1/8641 + 1/12664 + 1/18560 + 1/27201 + 1/39865 + 1/58425 + 1/85626 + 1/125491 + 1/183916 + 1/269542 + 1/395033 + 1/578949 + 1/848491 + 1/1243524 + 1/1822473 + 1/2670964 + 1/3914488 + 1/5736961 + 1/8407925 + \ldots = 4,60320706057253 \]

It is estimated that the value tends to

4,6033

This value divided by 7 gives 0,6576 value very near to the spin of the final black hole produced by the collision of two blacks holes and calculated from the observations of gravitational waves.

The number of elements \( N(x) \) less than or equal to \( x \) is given by the following formula:

\[ N(x) \leq \frac{1}{\ln \Psi} \ln(x\sqrt{5}) = 2,6161 \ln(x\sqrt{5}) \]

Where

\( \Psi = 1,46557123187676802665\ldots \) (supergolden ratio)

\( N(100) = 14 \) (calculated \( \approx 14,15 \))

\( N(1000) = 20 \) (calculated \( \approx 20,18 \))

The nth element is given by the following formula:

\[ x \approx \frac{1}{\sqrt{5}} e^{N} \]

\( N \)
\[ x(10000) = 5 \times 10^{1659} \text{ (calculated } = 5 \times 10^{1659}) \]
22. PARTITION OF A NUMBER

A partition of a positive integer is a way of writing $n$ as the sum of positive integers, without regard to the order of its parts. For example, 4 can be partitioned in five distinct ways:

1 4
2 3+1
3 2+2
4 2+1+1
5 1+1+1+1

The sum of the first 50 reciprocals is given by:

\[ S = 1 + 1 + 1/2 + 1/3 + 1/5 + 1/7 + 1/11 + 1/15 + 1/22 + 1/30 + 1/42 + 1/56 + 1/77 + 1/101 + 1/135 + 1/176 + 1/231 + 1/297 + 1/385 + 1/490 + 1/627 + 1/792 + 1/1002 + 1/1255 + 1/1575 + 1/1958 + 1/2436 + 1/3010 + 1/3718 + 1/4565 + 1/5604 + 1/6842 + 1/8349 + 1/10143 + 1/12310 + 1/14883 + 1/17977 + 1/21637 + 1/26015 + 1/31185 + 1/37338 + 1/44583 + 1/53174 + 1/63261 + 1/75175 + 1/89134 + 1/105558 + 1/124754 + 1/147273 + 1/173525 + \ldots = 3.51056310463079 \]

It is estimated that the value tends to

3,51061

This value divided by 4 gives 0,8776 value that is very near to the proton dimension.

This value divided by 8, that is also Fibonacci’s number and the number of the modes of the physical vibrations of the superstrings, gives 0,438820. The mean of the two values gives 0,65821 value very near to the spin of the final black hole produced by the
collision of two black holes and calculated from the observations of gravitational 
waves.

The \( x(n) \) element is given by the following approximate formula:

\[
x(n) \approx \frac{1}{4n\sqrt{3}} e^{\pi \sqrt{\frac{2n}{3}}} \text{ per } n \to \infty
\]

\[
x(10000) \approx 3.61673 \times 10^{106} \text{ (calculated } \approx 3.6328058 \times 10^{106})
\]

If we take the expression of Hardy-Ramanujan

\[
p(n) = \sum_{k=1}^{[\sqrt{n}]} P_k(n) + O(n^{-1/4}) = \frac{1}{2\pi \sqrt{2}} \sum_{k=1}^{[\sqrt{n}]} A_k(n) \sqrt{k} \frac{d}{dn} \left( \frac{\exp\left\{ \frac{\pi}{k} \sqrt{3} \left( \frac{n-1}{24} \right) \right\}}{\sqrt{n-\frac{1}{24}}} \right) + O(n^{-1/4})
\]

Now, we take the values for the number of partition of 15 and 21, i.e. \( p(15) = 176 \) and \( p(21) = 792 \), we obtain new interesting relationships. For \( p(15) = 176 \), we have:

\[
p(15) = \sum_{k=1}^{[\sqrt{15}]} P_k(n) + O(n^{-1/4}) = \frac{1}{2\pi \sqrt{2}} \sum_{k=1}^{[\sqrt{15}]} A_k(n) \sqrt{k} \frac{d}{dn} \left( \frac{\exp\left\{ \frac{\pi}{k} \sqrt{3} \left( \frac{n-1}{24} \right) \right\}}{\sqrt{n-\frac{1}{24}}} \right) + O(n^{-1/4}) = 176
\]

\[
p(15) = \sum_{k=1}^{[\sqrt{21}]} P_k(n) + O(n^{-1/4}) = \frac{1}{2\pi \sqrt{2}} \sum_{k=1}^{[\sqrt{21}]} A_k(n) \sqrt{k} \frac{d}{dn} \left( \frac{\exp\left\{ \frac{\pi}{k} \sqrt{3} \left( \frac{n-1}{24} \right) \right\}}{\sqrt{n-\frac{1}{24}}} \right) + O(n^{-1/4}) = 22 \times \frac{24}{3}
\]
24 = \frac{3}{22} \frac{1}{2\pi \sqrt{2}} \sum_{k=1}^{[\sqrt{n}]} A_k(n) k \frac{d}{dn} \left( \exp \left( \frac{\pi}{k} \sqrt{\frac{2(n-1)}{24}} \right) \right) + \frac{3}{22} O(n^{-1/4}).

But we know that with regard the number 24, it is related to the "modes" that correspond to the physical vibrations of the bosonic strings by the following Ramanujan function:

\begin{align*}
24 &= 4 \text{anti log} \left[ \int_{0}^{\infty} \cos \frac{\pi x w'}{\cosh \pi x} e^{-\pi i w'} dx \right] \sqrt{\frac{142}{t^2 w'}} \cdot \\
&\quad \log \left[ \sqrt{\frac{10 + 11\sqrt{2}}{4}} + \sqrt{\frac{10 + 7\sqrt{2}}{4}} \right].
\end{align*}

Thence, we have the following new mathematical connection also with the formula concerning the partition of a number:

24 = \frac{3}{22} \frac{1}{2\pi \sqrt{2}} \sum_{k=1}^{[\sqrt{n}]} A_k(n) k \frac{d}{dn} \left( \exp \left( \frac{\pi}{k} \sqrt{\frac{2(n-1)}{24}} \right) \right) + \frac{3}{22} O(n^{-1/4}).
For $p(21) = 792$, we have that

$$p(21) = \sum_{k=1}^{[\sqrt{21}]} P_1(n) + O(n^{-1/4}) = \frac{1}{2\pi \sqrt{2}} \sum_{k=1}^{[\sqrt{21}]} A_k(n) \sqrt{k} \frac{d}{dn} \left( \exp \left( \frac{\pi}{k} \sqrt{\frac{2(n-1)}{3}} \right) \right) + O(n^{-1/4}) = 792$$

and

$$p(21) = \sum_{k=1}^{[\sqrt{21}]} P_1(n) + O(n^{-1/4}) = \frac{1}{2\pi \sqrt{2}} \sum_{k=1}^{[\sqrt{21}]} A_k(n) \sqrt{k} \frac{d}{dn} \left( \exp \left( \frac{\pi}{k} \sqrt{\frac{2(n-1)}{3}} \right) \right) + O(n^{-1/4}) = 33 \times 24$$

$$24 = \frac{1}{33} \frac{1}{2\pi \sqrt{2}} \sum_{k=1}^{[\sqrt{21}]} A_k(n) \sqrt{k} \frac{d}{dn} \left( \exp \left( \frac{\pi}{k} \sqrt{\frac{2(n-1)}{3}} \right) \right) + \frac{1}{33} O(n^{-1/4})$$

Thence, for the Ramanujan’s modular function above described, we obtain the following new interesting formula:
24 = \frac{1}{33} \sum_{k=1}^{\lfloor e/\pi \rfloor} A_k(n)\sqrt{k} \frac{d}{dn} \left( \exp \left( \frac{\pi}{k} \sqrt{\frac{2}{3}} \left( \frac{n - 1}{24} \right) \right) \right) + \frac{1}{33} O(n^{-1/4}) =

\int_0^\infty \cos \left( \pi x w' \right) e^{-\pi^2 x^2} dx \frac{\sqrt{142}}{x^3 w'}

\log \left[ \sqrt{\left( \frac{10 + 11\sqrt{2}}{4} \right)} + \sqrt{\left( \frac{10 + 7\sqrt{2}}{4} \right)} \right].

With regard \( p(15) \) and \( p(21) \), we note also that 15 = 5*3 and 21, thence, mathematical connections with the Fibonacci's numbers 3, 5 and 21.

Thence, mathematical connections between partition numbers, size of the proton (that is a fermionic string) and spin number of the final black hole produced by the collision of two black holes and calculated from the observations of gravitational waves.
23.  PRONIC NUMBERS

A pronic number (also called oblong numbers, or heteromecic numbers or rectangular numbers) is a number that is the product of two consecutive numbers, that is, a number in the form \( n(n+1) \).

All pronic numbers are even (being the product of two consecutive numbers, of which at least one is even); 2 is also the only first number of this sequence, as well as the only one that is also a Fibonacci number.

The sum of the reciprocals is given by:

\[
S = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \frac{1}{72} + \frac{1}{90} + \frac{1}{110} + \frac{1}{132} + \frac{1}{156} + \frac{1}{182} + \frac{1}{210} + \frac{1}{240} + \frac{1}{272} + \frac{1}{306} + \ldots = 1
\]

The number of elements \( N(x) \) less than or equal to \( x \) is given by the following formula:

\[
N(x) \leq \frac{\sqrt{4x+1} - 1}{2}
\]

\( N(100) = 9 \) (calculated \( \approx 9.51 \))
\( N(1000) = 31 \) (calculated \( \approx 31.126 \))

The \( n \)th element is given by the inverse formula that’s the following:

\( x = n(n+1) \)

\( x(10000) = 100010000 \) (calculated \( = 100010000 \) is obviously the same value)
24. SUM-FREE SEQUENCE OF

A sum-free sequence is an increasing sequence of positive integers

\{n_k\}_{k \in \mathbb{N}}

such that for each k, \(n_k\) cannot be represented as a sum of any subset of the previous items the same sequence.

A classic example are the powers of 2:

1, 2, 4, 8, 16, ….

It forms a sum-free sequence because each element of the sequence is "1" more than the sum of all the previous elements, and therefore cannot represent the sum of the previous items.

We know that in this case the sum of the reciprocals is given by 2.

If R is the maximum value of a sequence of sums of reciprocal of any sequence of free sum, then it has been proven that the value of R is always less than:

\[ R < 3.0752 \]

For example the free sum of \{1, 2, 3, ..., n\} is given by

1, 2, 3, 6, 9, 16, 24, 42, 61, 108, 151, 253, 369, 607, 847, 1400, 1954, 3139, 4398, 6976, 9583, 15456, 20982, 32816, 45417, 70109, 94499, 148234, 200768, 308213, 415543, 634270, 849877, 1311244, 1739022, 2630061, 3540355, 5344961, 7051789, 10747207, 14158720, 21295570, 28188520, 42283059, 55560183, 83902379…….
The sum of the first 46 reciprocals is given by:

\[ S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{16} + \frac{1}{24} + \frac{1}{61} + \frac{1}{108} + \frac{1}{151} + \frac{1}{253} + \frac{1}{369} + \frac{1}{607} + \frac{1}{847} + \frac{1}{1400} + \frac{1}{1954} + \frac{1}{3139} + \frac{1}{4398} + \frac{1}{6976} + \frac{1}{9583} + \frac{1}{15456} + \frac{1}{20982} + \frac{1}{32816} + \frac{1}{45417} + \frac{1}{70109} + \frac{1}{94499} + \frac{1}{148234} + \frac{1}{200768} + \frac{1}{308213} + \frac{1}{415543} + \frac{1}{634270} + \frac{1}{849877} + \frac{1}{1311244} + \frac{1}{1739022} + \frac{1}{2630061} + \frac{1}{3540355} + \frac{1}{5344961} + \frac{1}{7051789} + \frac{1}{10747207} + \frac{1}{14158720} + \frac{1}{21295570} + \frac{1}{28188520} + \frac{1}{42283059} + \frac{1}{55560183} + \frac{1}{83902379} + \cdots = 2.283085362281 \]

It is estimated that the value tends to

2,28308541

We note that 2,28308541 – 1,618… = 0,665… value very near to the spin of the final black hole produced by the collision of two black holes and calculated from the observations of gravitational waves.
25. RAMANUJAN PRIME NUMBERS

In 1919, Ramanujan, Indian mathematician, published a new proof of Bertrand's postulate - which states that between a number \( n > 1 \) and its double exists at least one prime number. The result of Ramanujan is the following formula:

\[
\pi(x) - \pi(\frac{x}{2}) \geq 1, \, 2, \, 3, \, 4, \, 5, \, \ldots \quad \text{For all} \quad x \geq 2, \, 11, \, 17, \, 29, \, 41, \, \ldots
\]

where \( \pi(x) \) is the prime counting function, equal to the number of primes less than or equal to \( x \).

The case \( \pi(x) - \pi(\frac{x}{2}) \geq 1 \) for all \( x \geq 2 \) is the **Bertrand's postulate**.

We have so that the \( n \)th **prime Ramanujan** is the smallest number \( R_n \) such that

\[
\pi(x) - \pi(\frac{x}{2}) \geq n \quad \text{for all} \quad x \geq R_n.
\]

The series is given by the following primes:

\[
2, \, 11, \, 17, \, 29, \, 41, \, 47, \, 59, \, 67, \, 71, \, 97, \, 101, \, 107, \, 127, \, 149, \, 151, \, 167, \, 179, \, 181, \, 227, \, 229, \, 233, \, 239, \, 241, \, 263, \, 269, \, 281, \, 307, \, 311, \, 347, \, 349, \, 367, \, 373, \, 401, \, 409, \, 419, \, 431, \, 433, \, 439, \, 461, \, 487, \, 491, \, 503, \, 569, \, 571, \, 587, \, 593, \, 599, \, 601, \, 607, \, 641, \, 643, \, 647, \, 653, \, 659
\]

The sum of the first 73 reciprocals is given by:

\[
S = \frac{1}{2} + \frac{1}{11} + \frac{1}{17} + \frac{1}{29} + \frac{1}{41} + \frac{1}{47} + \frac{1}{59} + \frac{1}{67} + \frac{1}{71} + \frac{1}{97} + \frac{1}{101} + \frac{1}{107} + \frac{1}{127} + \frac{1}{149} + \frac{1}{151} + \frac{1}{167} + \frac{1}{179} + \frac{1}{181} + \frac{1}{227} + \frac{1}{229} + \frac{1}{233} + \frac{1}{239} + \frac{1}{241} + \frac{1}{263} + \frac{1}{269} + \frac{1}{281} + \frac{1}{307} + \frac{1}{311} + \frac{1}{347} + \frac{1}{349} + \frac{1}{367} + \frac{1}{373} + \frac{1}{401} + \frac{1}{409} + \frac{1}{419} + \frac{1}{431} + \frac{1}{433} + \frac{1}{439} + \frac{1}{461} + \frac{1}{487} + \frac{1}{491} + \frac{1}{503} + \frac{1}{569} + \frac{1}{571} + \frac{1}{587} + \frac{1}{593} + \frac{1}{599} + \frac{1}{601} + \frac{1}{607} + \frac{1}{641} + \frac{1}{643} + \frac{1}{647} + \frac{1}{653} + \frac{1}{659} + \frac{1}{677} + \frac{1}{719} + \frac{1}{727} + \frac{1}{739} + \frac{1}{751} + \frac{1}{769} + \frac{1}{809} + \frac{1}{821} + \frac{1}{823} + \frac{1}{827} + \frac{1}{853} + \frac{1}{857} + \frac{1}{881} + \frac{1}{937} + \frac{1}{941} + \frac{1}{947} + \frac{1}{967} + \frac{1}{983} + \frac{1}{1009} \ldots = 0.9586854078704516312243865479188
\]
It is estimated that the value tends to

\[ 1.6 \] value very near to the aurea ratio 1.618

The 10000\(n\)th element is:

\[ x(10000) = 242057 \]
26. SOPHIE GERMAIN PRIME NUMBERS

A prime number of Sophie Germain, French mathematician, is a prime number p such that 2p + 1 is also a prime number.

The series is given by the following primes:


Of course, no Sophie Germain prime can have as the last digit "7".

The sum of the first 61 reciprocals is given by:

\[ S = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{11} + \frac{1}{23} + \frac{1}{29} + \frac{1}{41} + \frac{1}{53} + \frac{1}{83} + \frac{1}{89} + \frac{1}{113} + \frac{1}{131} + \frac{1}{173} + \frac{1}{179} + \frac{1}{191} + \frac{1}{233} + \frac{1}{239} + \frac{1}{251} + \frac{1}{281} + \frac{1}{293} + \frac{1}{359} + \frac{1}{419} + \frac{1}{431} + \frac{1}{443} + \frac{1}{491} + \frac{1}{509} + \frac{1}{593} + \frac{1}{641} + \frac{1}{653} + \frac{1}{659} + \frac{1}{683} + \frac{1}{719} + \frac{1}{743} + \frac{1}{761} + \frac{1}{809} + \frac{1}{911} + \frac{1}{953} + \frac{1}{1013} + \frac{1}{1049} + \frac{1}{1093} + \frac{1}{1223} + \frac{1}{1229} + \frac{1}{1289} + \frac{1}{1409} + \frac{1}{1439} + \frac{1}{1451} + \frac{1}{1481} + \frac{1}{1499} + \frac{1}{1511} + \frac{1}{1559} + \frac{1}{1583} + \frac{1}{1601} + \frac{1}{1733} + \frac{1}{1811} + \frac{1}{1889} + \frac{1}{1901} + \frac{1}{1931} + \frac{1}{1973} + \frac{1}{2003} \cdots = \frac{1,3671171856607302530684755842884}{2} \]

The value \(1,367\ldots\) divided by 2 gives 0.6835 value very near to the spin of the final black hole produced by the collision of two black holes and calculated from the observations of gravitational waves.

It is estimated that the value tends to

\[ 1.54 \] that is near to the value of the aurea ratio, i.e. 1.618…
The number of elements \( N(x) \) less than or equal to \( x \) is given by the following formula:

\[
N(x) \leq 2 \cdot C_2 \frac{x}{(\ln x)^2} = 1,3203 \frac{x}{(\ln x)^2}
\]

where \( C_2 \) is the twin prime = 0.6601611815

\[
N(100) = 10 \quad \text{(calculated } \approx 6.225) \\
N(1000) = 37 \quad \text{(calculated } \approx 27,67)
\]

The \( n \)th element is given by the inverse formula that’s the following:

\[
x \approx 1,3203 \cdot N(\ln N)^2
\]

\[
x(10000) = 1349363 \quad \text{(calculated } \approx 1120015,37)
\]
27. **TABLE ORDERED ACCORDING TO THE INCREASING VALUE OF SUM OF RECIPROCAL**

**TAB. 2**

<table>
<thead>
<tr>
<th>NUMBERS</th>
<th>VALUE</th>
<th>10000° Element</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kempner series of 9 modified harmonic series, formed by omitting 9</td>
<td>22,920676619264</td>
<td>15553</td>
</tr>
<tr>
<td>Fibonacci cows sequence</td>
<td>4,6033</td>
<td>1E+1659</td>
</tr>
<tr>
<td>Partition number (a way of writing n as a sum of positive integers)</td>
<td>3,51061</td>
<td>1E+106</td>
</tr>
<tr>
<td>Palindromic number</td>
<td>3,370283259497</td>
<td>8999998</td>
</tr>
<tr>
<td>Fibonacci numbers</td>
<td>3,359885666243</td>
<td>1E+2089</td>
</tr>
<tr>
<td>Factorial</td>
<td>2,718281828459</td>
<td>1E+35659</td>
</tr>
</tbody>
</table>
| **Fibonorial or Fibonacci factorial**

\[ n!F = \prod_{i} F_{i} \]

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>1E+10446932</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of sum-free subsets of {1, 2, 3, ..., n}</td>
<td>2,28308541</td>
<td>?</td>
</tr>
<tr>
<td>Sexy primes</td>
<td>S &gt; 2</td>
<td>554893</td>
</tr>
<tr>
<td>Powers of two 2^n</td>
<td>2</td>
<td>1E+3009</td>
</tr>
<tr>
<td>Twin primes</td>
<td>1,902160583104</td>
<td>1260989</td>
</tr>
<tr>
<td>Cousin primes</td>
<td>1,673235376190</td>
<td>1266487</td>
</tr>
<tr>
<td>Square numbers (the Basel problem)</td>
<td>1,644934066848</td>
<td>100000000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>--------------------------</td>
<td>--------------------------</td>
<td>--------------------------</td>
</tr>
<tr>
<td><strong>Exponential factorial</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a(1)=1$, $a(n+1) = (n+1)^{a(n)}$</td>
<td>$x^{(x-1)^{(x-2)^\ldots}}$</td>
<td>$1,611114925808$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$10000^{9999^{9998^{\ldots}}}$</td>
</tr>
<tr>
<td><strong>Ramanujan prime</strong></td>
<td>$\pi(x) - \pi(x/2) \geq 1, 2, 3, 4, 5, \ldots$ per tutti gli $x \geq 2, 11, 17, 29, 41, \ldots$</td>
<td>$1,6$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Sophie Germain prime</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p$ and $2p+1$ are both primes</td>
<td></td>
<td>$1,54$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Palindromic prime</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1,323982146806$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$13649694631$</td>
</tr>
<tr>
<td><strong>Heptagonal numbers</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1,304763183779$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$249985000$</td>
</tr>
<tr>
<td><strong>Cubes of positive integers</strong></td>
<td></td>
<td>$1,202056903160$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1E+12$</td>
</tr>
<tr>
<td><strong>Fibonacci primes</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1,126447227672$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1E+(1E+1341)$</td>
</tr>
<tr>
<td><strong>Perfect power with duplications</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$87403801$</td>
</tr>
<tr>
<td><strong>Perfect power $p-1$ without duplications</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$90706575$</td>
</tr>
<tr>
<td><strong>Pronic number</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$100010000$</td>
</tr>
<tr>
<td><strong>Perfect power without duplications</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0,874464368$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$90706576$</td>
</tr>
<tr>
<td><strong>Primorial Factorial</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0,705230171792$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1E+40400$</td>
</tr>
<tr>
<td><strong>Fermat number</strong></td>
<td>$F(n) = 2^{(2^n)} + 1$ tutti dispari coprimi</td>
<td>$0,596063172118$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1E+(1E+4341)$</td>
</tr>
</tbody>
</table>

Note: in orange in the 2nd column, the estimated values of the sum of reciprocals.
28. **OBSERVATIONS**

The sum of the reciprocals gives fundamental information on these series. The higher this sum and larger fractional numbers there are in series and vice versa.

They are all positive series. They are all convergent, and their values can be integer, rational or irrational (algebraic or transcendental).

If we look at the values of the sums of the reciprocals we have only 3 series with values equal to 1:

<table>
<thead>
<tr>
<th>Series Description</th>
<th>Sum of Reciprocals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perfect power with duplications</td>
<td>$1/4+1/8+1/9+1/16+1/16+1/25+1/27+1/32+1/36+1/49+1/64+1/64+1/64+1/81+1/81+1/100$</td>
</tr>
<tr>
<td>Perfect power $p-1$ without duplications</td>
<td>$1/3+1/7+1/8+1/15+1/24+1/26+1/31+1/35+1/48+1/63+1/80+1/99$</td>
</tr>
<tr>
<td>Pronic number</td>
<td>$1/2+1/6+1/12+1/20+1/30+1/42+1/56+1/72+1/90+1/110+1/132+1/156+1/182+1/210+1/240$</td>
</tr>
</tbody>
</table>

This means that these 3 series have the same the sums of the reciprocals and we can call them “congruent” as in geometry:

1. $S=1/4+1/8+1/9+1/16+1/16+1/25+1/27+1/32+1/36+1/49+1/64+1/64+1/64+1/81+1/81+1/100+\ldots=1$
2. $S=1/3+1/7+1/8+1/15+1/24+1/26+1/31+1/35+1/48+1/63+1/80+1/99+\ldots=1$
3. $S=1/2+1/6+1/12+1/20+1/30+1/42+1/56+1/72+1/90+1/110+1/132+1/156+1/182+1/210+1/240+1/272+1/306+\ldots=1$
For the first two we have the equivalence:

\[
\sum_{p} \frac{1}{p-1} = \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{m^n} = 1
\]

The sum of the reciprocals of the third series - pronic numbers - is a telescoping series that sums to 1:

\[
1 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots = \sum_{i=1}^{\infty} \frac{1}{i(i+1)}
\]

The partial sum of the first \(n\) terms in this series is

\[
\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}
\]

Of course, observing only the sums of the infinite fractions is virtually impossible to establish that they have the same sums but it's still amazing that instead their sums to infinity are equal.

Each series converges to a value that is unique and characteristic of the entire series.

Then each value of the sum of the reciprocals is tied to a particular series.

We have seen that if the series have the same values of convergence, thus the sums of their fractional infinite terms are not equal: we have “congruent” series.

So for each positive integer, rational or real number may also be associated more different series and then not worth the vice versa.

Furthermore we understand also what is the growth factor of the series and that there is a clear link between the sums of the reciprocal and the "intrinsic nature" of the series.
29. Note 1

Black strings as black hole solutions that arise in type II string theory

We have the following action:

\[
S = \int d^{10}x \sqrt{-g} \left[ e^{-2\phi} \left( R + 4(\nabla \phi)^2 \right) - \frac{2e^{2\alpha\phi}}{(D-2)} F^2 \right], \tag{1}
\]

where \( F \) is a \( D-2 \) form satisfying \( dF = 0 \). We will assume \( D \geq 4 \). We require translational and rotational symmetry in \((10-D)\) dimensions. The most general ten-dimensional metric with the prescribed symmetry is

\[
ds^2 = e^A ds^2 + e^B dx_i dx_j, \tag{2}\]

where \( ds^2 \) is an arbitrary D-dimensional Lorentzian metric, \( x^i \) are \((10-D)\) Cartesian coordinates, and all fields are independent of \( x^i \).

We wish to define \( A, B \) and \( \phi \) as linear combinations of two scalar fields \( \rho \) and \( \sigma \) so that the kinetic term for \( \hat{g} \) is the standard Einstein action, \( \rho \) and \( \sigma \) are canonically normalized, and \( \sigma \) does not couple to \( F^2 \). This is accomplished by the definitions

\[
\beta\phi = \rho \frac{(4\alpha + 7 - D)}{2} - \sigma \frac{(D-3)}{2} \left( \frac{10-D}{D-2} \right)^{1/2},
\]
\[
\beta A = \rho \left[ \frac{\alpha - \frac{D-4}{D-2}}{2} \right] - \sigma (\alpha + 1) \left( \frac{10-D}{D-2} \right)^{1/2},
\]
\[
\beta B = \rho (\alpha + 1) + \sigma \frac{(D-2)\alpha - D + 4}{\left[ (10-D)(D-2) \right]^{1/2}} \tag{3}\]

where \( \beta \) is given by

\[
\beta = \left[ 4\alpha^2 + 2\alpha(7-D) + 2\frac{D-1}{D-2} \right]^{1/2} \tag{4}\]

The expression for the charged black hole in \( D \) dimensions is:
where the exponent is

$$\gamma = \frac{2\beta^2(D-2)}{(D-3)(2(D-3)+\beta^2(D-2))}, \quad (6)$$

and the charge $Q$ is given by

$$Q = \sqrt{\frac{\gamma(D-3)(r_\gamma r)^{D-3}}{2\beta^2}}, \quad (7)$$

Using (2), (3) and (5), one obtains black $(10 - D)$-brane solutions of (1):

$$F = Q_{\epsilon_{\alpha-2}},$$

$$ds^2 = \left[1 - \left(\frac{r_\epsilon}{r}\right)^{D-3}\right] \left[1 - \left(\frac{r_\epsilon}{r}\right)^{D-3}\gamma_r^2\right] dt^2 + \left[1 - \left(\frac{r_\epsilon}{r}\right)^{D-3}\gamma_r^2\right] dr^2 +$$

$$+ r^2 \left[1 - \left(\frac{r_\epsilon}{r}\right)^{D-3}\right] d\Omega_{d-2}^2 + \left[1 - \left(\frac{r_\epsilon}{r}\right)^{D-3}\gamma_r^2\right] dx^i dx_i,$$

$$e^{-2\phi} = \left[1 - \left(\frac{r_\epsilon}{r}\right)^{D-3}\right]^{\gamma_r}, \quad (8)$$

where the exponents are given by
\[ \gamma_r = \delta(\alpha-1) - \frac{D-5}{D-3}, \quad \gamma_s = \delta(\alpha+1), \quad \gamma_\phi = -\delta(4\alpha + 7 - D) \quad (9) \]

and

\[ \delta = \left(2\alpha^2 + (7-D)\alpha + 2\right)^{-1}. \quad (10) \]

To obtain black p-branes with “electric” charges, we dualize these forms. We start with the action (1). The equation of motion for \( F \) is \( \nabla^\mu \left( e^{2\alpha \phi} F_{\mu \nu} \right) = 0 \). This implies that the \( 12-D \) form

\[ K = e^{2\alpha \phi} \ast F \quad (11) \]

is closed, where \( \ast \) denotes the Hodge dual and obeys \( \ast^2 = 1 \). If we replace \( F \) with \( K \) in the equations of motion that follow form (1), we obtain a set of equations which can be derived from the action (1) with \( F^2 \) replaced by \( K^2 \) and \( \alpha \) replaced by \( -\alpha \). Thus to obtain the dual of a solution with parameters \( \alpha, D \), we make the transformation \( \alpha \rightarrow -\alpha \) and \( D \rightarrow 14-D \). It is amusing to note that in this sense 10 and 4 dimensions are dual.

If we dualize the three-form \( H \), we get the solution with \( D=9 \) and \( \alpha=1 \). This has exponents

\[ \gamma_r = -\frac{2}{3} = 0.66666666... \approx 0.67; \quad \gamma_s = 1, \quad \gamma_\phi = -1. \quad (12) \]

We note that the value of \( \gamma_r \) is very near to the constant of the twin primes (0.6601611815) and to the value of the spin of the final black hole produced by the collision of two black holes and calculated from the observations of gravitational waves.

In this case, the fractional exponents can be removed by introducing a new radial coordinate \( y^6 = r^6 - r_6^6 \). The solution then becomes
\[ ds^2 = -\frac{1-C}{y^6} dt^2 + \frac{dx^2}{1+r^6 y^6} + \frac{dy^2}{1-r^6 y^6} + y^2 d\Omega^2, \quad e^{-2\phi} = 1 + \frac{r^6}{y^6}, \quad H = Qe^{2\phi} \epsilon, \quad (13) \]

where \( C = r_+^6 - r_-^6 \). This represents a family of axion string solutions surrounded by event horizons i.e. black strings. At the extremal value, \( C = 0 \), the event horizon becomes singular.

**Conclusions**

It is interesting that almost all numbers analyzed provide the values 0.67 and 0.84 thence values very near, respectively, to the spin of the final black hole produced by the collision of two black holes and calculated from observations of gravitational waves and the size of a proton. This may be a further evidence that the mathematical constants are always present in Nature. No coincidence that the constant value of the twin primes \( = 0.6601611815 \) is practically very near to the spin of the black hole before mentioned.

**30. Appendix A**

**Final reference to the constant \( \frac{1}{2} \) for the Riemann zeta function**

We can consider the real part \( \frac{1}{2} \) of the Riemann zeta function as a constant for all the infinite zeros, as **arithmetic mean** between two conjugate zeros that make up any zero of zeta, from the smallest to the largest ever to have been calculated; as well as \( \pi = 1.314 \) is always the ratio between circumference and diameter for all possible circles with a diameter of one millimeter to that, for example, by a billion kilometers, which, obviously is not the largest of all.

Such mean and related conjecture is described and calculated in Ref. 1 (In English), 2
and 3 (in Italian) on our conjecture of generalized zeta functions. As sum of the reciprocal famous numbers, $\frac{1}{2}$ can be considered as half of the sum of the reciprocals of the numbers powers perfect with duplications.

From Rif. _SOMMA DEI RECIPROCI DI NUMERI FAMOSI_ Ing. Pier Franz Roggero, Dott. Michele Nardelli, P.A. Francesco Di Noto - Published in italian on our site:

7. PERFECT POWER WITH DUPLICATIONS

A perfect power is a positive integer that can be expressed as a power of another positive integer. More formally, $n$ is a perfect power if there are natural numbers $m > 1$ and $k > 1$ such that $n = m^k$. In the case in which $k = 2$ will have the perfect squares in the case of $k = 3$ you will have the perfect cubes. The number 1 in general is not considered (because $1^k = 1$ for any $k$).

The sum of the reciprocals with duplication is given by:

$$S = \frac{1}{4} + \frac{1}{8} + \frac{1}{9} + \frac{1}{16} + \frac{1}{16} + \frac{1}{25} + \frac{1}{27} + \frac{1}{32} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{81} + \frac{1}{81} + \frac{1}{100} + \cdots = 1$$

Then $1: 2 = 0.5 = 1/2 = $ also real part of the zeros of the zeta function

But also, likewise

8. PERFECT POWER P-1 WITHOUT DUPLICATIONS

Euler and Goldbach have shown that the sum of the reciprocals of $1 / p - 1$ excluding the value 1 and without duplication is given by:

$$S = \frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \frac{1}{31} + \frac{1}{35} + \frac{1}{48} + \frac{1}{63} + \frac{1}{80} + \frac{1}{99} + \cdots$$

Then, also in this case $1: 2 = 0.5 = 1/2 = $ also real part of the zeros of the zeta function
23. PRONIC NUMBERS

A pronic number (or oblong number or also eteromecic number) is a number that is the product of two consecutive numbers, that is, a number in the form \( n(n+1) \).
All the pronic numbers are even (being the product of two consecutive numbers, of which at least one is even); 2 is also the only first number of this sequence, as well as the only that is also a Fibonacci’s number.
The sum of the reciprocals is given by:
\[
S = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \frac{1}{72} + \frac{1}{90} + \frac{1}{110} + \frac{1}{132} + \frac{1}{156} + \frac{1}{182} + \frac{1}{210} + \frac{1}{240} + \frac{1}{272} + \frac{1}{306} + \ldots = 1
\]
Then \( 1: 2 = 0.5 = 1/2 = \) also real part of the zeros of the zeta function

Or, equivalently:

14. POWERS OF 2

A power of two is any whole number power of the number two, or that it can get by multiplying two by itself a certain number of times. A power of two is also 1, as \( 2^0 = 1 \).
The sum of the reciprocals is given by:
\[
S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} + \ldots = 2
\]
In this case \( \frac{1}{2} = 2/4 = 0.5 = 1/2 \)

The sum, however, very near to the zeta function is that relative to the POWER PERFECT WITH DUPLICATIONS
\[
S = \frac{1}{4} + \frac{1}{8} + \frac{1}{9} + \frac{1}{16} + \frac{1}{16} + \frac{1}{25} + \frac{1}{32} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{81} + \frac{1}{81} + \frac{1}{100} + \ldots = 1
\]
Because regards all the powers of the natural numbers, including primes, although here without exponent \( z \) complex, that from the complex parts of the conjugates zeros of zeta, that in the mean cancel each other out, being opposites, and leave
only the real mean \( \frac{1}{2} \), see Ref. a, b and c

a) CONJECTURE ON ZETA FUNCTIONS GENERALIZED

Michele Nardelli, Francesco Di Noto, Pierfrancesco Roggero

b) RETTE CRITICHE LEGATE A \( \frac{1}{2} \) (Tramite l’ex congettura forte di Goldbach) - Ing. Pier Franz Roggero, Dott. Michele Nardelli, P.A. Francesco Di Noto

c) SECONDA PARTE DELLA CONGETTURA SULLE FUNZIONI ZETA GENERALIZZATE (Tabelle e grafici con nuovi indizi compatibili con la congettura) - Francesco Di Noto, Michele Nardelli, Pierfrancesco Roggero
31. REFERENCES

1) Wikipedia

2) Mathworld

3) I NUMERI FIBONORIALI F!(n), 2° PARTE - In this paper we show other connections between fibonorial numbers factors, their exponents, and other

4) TEORIA MATEMATICA DEI NODI, FISICA QUANTISTICA, TEORIA DI STRINGA (connessioni con i numeri di Fibonacci, di Lie e i numeri di partizione) - Francesco Di Noto, Michele Nardelli, Pierfrancesco Roggero - In questo lavoro mostriamo qualche possibile relazione tra la teoria di stringa e teoria matematica dei nodi, tramite la comune connessione con i numeri di Fibonacci, di Lie e i numeri di partizione.

5) Properties of the binary black hole merger GW150914 - The LIGO Scientific Collaboration and The Virgo Collaboration (compiled 11 February 2016)

Abstract
On September 14, 2015, the Laser Interferometer Gravitational-wave Observatory (LIGO) detected a gravitational-wave transient (GW150914); we characterise the properties of the source and its parameters. The data around the time of the event were analysed coherently across the LIGO network using a suite of accurate waveform models that describe gravitational waves from a compact binary system in general relativity. GW150914 was produced by a nearly equal mass binary black hole of masses $36_{-3}^{+5}M_{\odot}$ and $29_{-4}^{+4}M_{\odot}$ (for each parameter we report the median value and the range of the 90% credible interval). The dimensionless spin magnitude of the more massive black hole is bound to be $< 0.7$ (at 90% probability). The luminosity distance to the source is $410_{-180}^{+160}$ Mpc corresponding to a redshift $0.09_{-0.04}^{+0.03}$ assuming standard cosmology. The source location
is constrained to an annulus section of 590 deg$^2$, primarily in the southern hemisphere. The binary merges into a black hole of mass $62_{-4}^{+4} M$ and spin $0.67_{-0.05}^{+0.05}$. This black hole is significantly more massive than any other known in the stellar-mass regime.

6) The size of the proton - Vol 466|8 July 2010| doi:10.1038/nature09250

Abstract:
On the basis of present calculations of fine and hyperfine splittings and QED terms, we find $r_p = 0.84184(67)$ fm, which differs by 5.0 standard deviations from the CODATA value of 0.8768(69) fm. This value is based mainly on precision spectroscopy of atomic hydrogen and calculations of bound-state quantum electrodynamics (QED; refs 8, 9). The accuracy of $r_p$ as deduced from electron–proton scattering limits the testing of bound-state QED in atomic hydrogen as well as the determination of the Rydberg constant (currently the most accurately measured fundamental physical constant).

7) Gary T. Horowitz and Andrew Strominger: “Black Strings and P-Branes”; Department of Physics University of California Apr-17-1991