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INVITATION TO THE "SPOOKY" QUANTUM PHASE-LOCKING EFFECT AND ITS LINK TO $1/F$ FLUCTUATIONS

MICHEL PLANAT *
Laboratoire de Physique et Métrologie des Oscillateurs du CNRS,
32 Avenue de l'observatoire, 25044 Besançon Cedex, France
planat@lpmo.edu

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An overview of the concept of phase-locking at the non linear, geometric and quantum level is attempted, in relation to finite resolution measurements in a communication receiver and its $1/f$ noise. Sine functions, automorphic functions and cyclotomic arithmetic are respectively used as the relevant trigonometric tools. The common point of the three topics is found to be the Mangoldt function of prime number theory as the generator of low frequency noise in the coupling coefficient, the scattering coefficient and in quantum critical statistical states. Huyghens coupled pendulums, the Adler equation, the Arnold map, continued fraction expansions, discrete Möbius transformations, Ford circles, coherent and squeezed phase states, Ramanujan sums, the Riemann zeta function and Bost and Connes KMS states are some but a few concepts which are used synchronously in the paper.

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1. Introduction

The interleaving of frequencies and phases of electronic oscillators interacting in non linear circuits follows arithmetical rules. Continued fraction expansions, prime number decompositions and related number theoretical concepts were successfully used to account for the experimental effects in mixers and phase-locked loops [1,2]. We also made use of these tools within the field of quantum optics emphasizing the hidden connection between phase-locking and cyclotomy [3]. As a matter of fact the understanding of phase effects in devices is so much entangled that it will reveal useful to check several clues relying on differential and discrete non linear equations, Fourier analysis, hyperbolic geometry and quantum mechanics. But over all the fundamental issue is the modelling of finite resolution measurements

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and how it puts a constraint on the performance, how it generates the fluctuations in oscillating circuits.

In Sect. 2 we report on the early history of phase-locking about classical observations of coupled mechanical pendulums and electronic oscillators. Non linear continuous and discrete generic models are introduced emphasizing the case of homodyne detection in a phase sensitive communication receiver. It is shown that low frequency noise happens due to low pass filtering and the finite resolution of frequency counts. A phenomenological model relating the coupling coefficient to prime numbers is developed. In Sec 3 the justification of the model is given by using the hyperbolic geometry of the half-plane applied to the low pass filtering. In Sec 4 the phase itself is taken to be discrete. Discrete quantum optics is related to quantum phase-locking and the arithmetic of $1/f$ noise.

2. Classical Phase-Locking: from Huyghens to the Prime Numbers

Being obliged to stay in my room for several days and also occupied in making observations on my two newly made clocks, I have noticed a remarkable effect which no one could have ever thought of. It is that these two clocks hanging next to one another separated by one or two feet keep an agreement so exact that the pendulums invariably oscillate together without variation. After admiring this for a while, I finally figured out that it occurs through a kind of sympathy: mixing up the swings of the pendulums, I have found that within a half hour always return to consonance and remain so constantly afterwards as long as I let them go. I then separated them, hanging one at the end of the room and the other fifteen feet away, and noticed that in a day there was five seconds difference between them. Consequently, their earlier agreement must in my opinion have been caused by an imperceptible agitation of the air produced by the motion of the pendulums.

The citation is taken from [4]. The authors remind a later letter by Huyghens that the coupling mechanism was in fact a small vibration transmitted through the wall, and not movement of air:

Lord Rayleigh (1907) made similar observations about two driven tuning forks coupled by vibrations transmitted through the table on which both forks sat... Locking in triode circuits was explained by Van der Pol (1927) who included in the equation for the triode oscillator an external electromotive force as given in

$$\frac{d^2v}{dt^2} - \frac{d}{dt}(gv - \beta' v^3) + \omega^2 v = \omega_0^2 V_0 \sin \omega_0 t, \tag{1}$$

where g is the linear net gain (i.e. the gain in excess of losses, β' the saturation coefficient, and ω is the resonance frequency in the absence of dissipation or gain. He showed that when an external electromotive force is included, of frequency ω_0, and tuned close to the oscillator frequency ω, the oscillator suddenly jumped to the external frequency. It is important to note that the beat note between the two frequencies vanishes not because the two frequencies vanish, not because the triode stops oscillating, but because it oscillates at the external frequency.

We can show the locking effect by utilizing the slowly varying amplitude approach, including a slowly varying phase Φ and oscillation at the external frequency ω_0 and *amplitude V*

$$\frac{d\Phi}{dt} + K \sin\Phi = \omega - \omega_0 = \omega_{LF}, \tag{2}$$

where we use ω_{LF} for the detuning term and $K = \omega_0 V_0/V$ for the locking coefficient [4].

The regime just described is the so-called injection locking regime, also found in injection-locked lasers. The equation (2) is the so-called Adler's equation of electronics [5].

One way to synthesize (2) is thanks to the phase-locked loop of a communication receiver. The receiver is designed to compare the information carrying external oscillator (RF) to a local oscillator (LO) of about the same high frequency through a non linear mixing element. For narrow band demodulation one uses a discriminator of which the role is first to differentiate the signal, that is to convert frequency modulation (FM) to amplitude modulation (AM) and second to detect its low frequency envelope: this is called baseband filtering. For more general FM demodulation one uses a low pass filter instead of the discriminator to remove the high frequency signals generated after the mixer. In the closed loop operation a voltage controlled LO (or VCO) is used to track the frequency of the RF. Phase modulation is frequently used for digital signals because low bit error rates can be obtained despite poor signal to noise ratio in comparison to frequency modulation [6].

Let us consider a type of receiver which consists in a mixer, in the form of a balanced Schottky diode bridge and a low pass filter. If f_0 and f are the frequencies of the RF and the LO, and $\theta(t)$ and $\psi(t)$ their respective phases, the set mixer and filter essentially behaves as a phase detector of sensitivity u_0 (in Volts/rad.), that is the instantaneous voltage at the output is the sine of the phase difference at the inputs

$$u(t) = u_0 \sin(\theta(t) - \psi(t)). \tag{3}$$

The non linear dynamics of the set-up in the closed loop configuration is well described by introducing the phase difference $\Phi(t) = sin(\theta(t) - \psi(t))$. Using $\dot{\theta} = \omega_0$ and $\dot{\psi(t)} = \omega + Au(t)$, with $\omega_0 = 2\pi f_0$, $\omega = 2\pi f$ and A (in rad. Hz/Volt) as the sensitivity of the VCO, one recovers Adler's equation (2) with the open loop gain $K = u_0 A$.

Equation (2) is integrable but its solution looks complex [7]. If the frequency shift ω_{LF} does not exceed the open loop gain K, the average frequency $\langle\dot{\Phi}\rangle$ vanishes after a finite time and reaches the stable steady state $\Phi(\infty) = 2l\pi + \sin^{-1}(\omega_{LF}/K)$, l integer. In this phase-tracking range of with $2K$ the RF and the LO oscillators are also frequency-locked. Outside the mode-locking zone there is a sech shape beat signal of frequency

$$\tilde{\omega}_{LF} = \langle\dot{\Phi}(t)\rangle = (\omega_{LF}^2 - K^2)^{1/2}. \tag{4}$$

The sech shape signal and the non linear dependance on parameters ω_{LF} and K are actually found in experiments [2, 7]. In addition the frequency ω_{LF} is fluctuating (see Fig. 1). It can be characterized by the Allan variance $\sigma^2(\tau)$ which is the mean squared value of the relative frequency deviation between adjacent samples in the time series, averaged over an integration time τ. Close to the phase-locked zone the Allan deviation is

$$\sigma(\tau) = \frac{\sigma_0 K}{\tilde{\omega}_{LF}}, \tag{5}$$

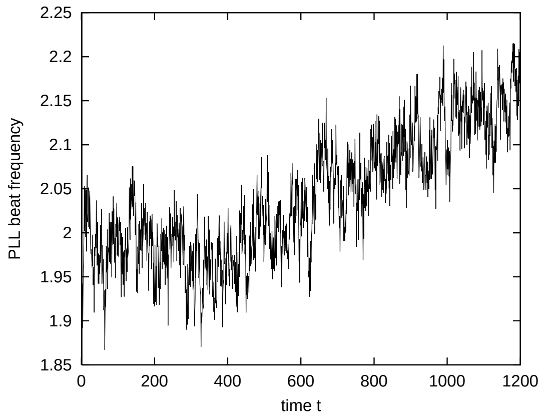

Fig 1. Fluctuating counts of the beat frequency (in Hz) close to the phase-locked zone. The inputs are quartz oscillators at 10 MHz. The power spectrum has a pure $1/f$ dependance

where σ_0 is a residual frequency deviation depending of the quality of input oscillators and that of the phase detector. Allan deviation is found independent of τ which is a signature of a $1/f$ frequency noise of power spectral density $S(f) = \sigma/(2 \ln 2f)$. One way to predict the dependence (5) is to use differentiation of (4) with respect to the frequency shift $\tilde{\omega_{LF}}$ so that

$$\delta\tilde{\omega}_{LF} = \delta\omega_{LF}(1 + K^2/\tilde{\omega}_{LF}^2)^{1/2}. \tag{6}$$

Relation (6) is defined outside the mode-locked zone $|\omega_{LF}| > K$; close to it, if the effective beat note $\tilde{\omega}_{LF} \le K$, the square root term is about $K/\tilde{\omega}_{LF}$. If one identifies $\delta\omega_{LF}/\tilde{\omega}_{LF}$ with a pedestal Allan deviation σ_0 and $\delta\tilde{\omega}_{LF}/\tilde{\omega}_{LF}$ with a magnified Allan deviation σ one explains the experimental result (5). One can conclude that, either the PLL set-up behaves as a microscope of an underlying flicker floor σ_0, or the $1/f$ noise is some dynamical property of the PLL. In the past we looked at a possible low dimensional structure of the time series and found a stable embedding dimension lower or equal to 4 [8]. But at that time the dynamical model of $1/f$ noise still remained elusive.

Adler's model presupposes a fundamental interaction $\omega_{LF} = |\omega_0 - \omega(t)|$ in the mixing of the two input oscillators. But the practical operation of the phase detector involves harmonic interactions of the form $\omega_{LF} = |p\omega_0 - q\omega(t)| \le \omega_c = 2\pi f_c$, where p and q are integers and f_c is the cut-off frequency of the low pass filter. This can be rewritten by introducing the frequency ratios $\nu = \frac{\omega(t)}{\omega_0}$ and $\mu = \frac{\omega_{LF}}{\omega_0}$ as $\mu = q|\nu - \frac{p}{q}|$. This form suggests that the aim of the receiver is to select such couples (p, q) which realize a "good" approximation of the "real" number ν. There is a mathematical concept which precisely does that: the diophantine approximator. It selects such couples p_i and q_i, coprime to each other, i.e. with greatest common divisor $(p_i, q_i) = 1$ from the continued fraction expansion of ν

$$\nu = [a_0; a_1, a_2, \cdots a_i] = a_0 + 1/(a_1 + 1/(a_2 + 1/\cdots + 1/(a_i \cdots))) \simeq \frac{p_i}{q_i}. \tag{7}$$

The diophantine approximation satisfies

$$|\nu - \frac{p_i}{q_i}| \leq \frac{1}{a_{i+1}q_i^2}. \quad (8)$$

The fraction $\frac{p_i}{q_i}$ is a so-called convergent and the a_i''s are called partial quotients. The approximation is truncated at the index i just before the partial quotient a_{i+1}. It should be observed that diophantine approximations are different from decimal approximations $\frac{c_i}{d_i}$ for which one gets $|\nu - \frac{c_i}{d_i}| \leq \frac{1}{d_i}$. It was shown [1] using the filtering condition that a_{i+1} identifies with a very simple expression

$$a_{i+1} = \left[\frac{f_0}{f_c q_i}\right], \quad (9)$$

where [ ] denotes the integer part. For example if one chooses $f_0 = 10$ MHz and $f_c = 300$ kHz, the fundamental basin $\frac{p_i}{q_i} = \frac{1}{1}$ will be truncated if $a_{i+1} \geq 33$ and the basin $\frac{p_i}{q_i} = \frac{3}{5}$ will be truncated if $a_{i+1} \geq 6$. The resulting full spectrum is a superposition of V-shape basins of which the edges are located at

$$\begin{aligned} \nu_1 &= \{a_0; , a_1, a_2, \cdots, a_i, a_{i+1}\}, \\ \nu_2 &= \{a_0; a_1, a_2, \cdots, a_{i-1}, 1, a_{i+1}\}, \end{aligned} \quad (10)$$

where the partial expansion before a_{i+1} corresponds to the two possible continued fractions of the rational number $\frac{p_i}{q_i}$. The basin of number $\nu = \frac{3}{5} = \{0; 1, 1, 2\}$ extends to $\nu_1 = \{0; 1, 1, 2, 33\} = \frac{19}{32} \simeq 0.594$, $\nu_2 = \{0; 1, 1, 1, 1, 33\} = \frac{31}{34} \simeq 0.618$. For a reference oscillator with $f_0 = 10$ MHz this corresponds to a frequency bandwidth $(0.618 - 0.594).10^7$ MHZ=240 kHz.

With these arithmetical rules in mind one can now tackle the difficult task to account for phase-locking of the whole set of harmonics. The differential equation for the phase shift $\dot{\Phi}(t; q_i, p_i)$ at the harmonic (p_i, q_i) corresponding to the beat frequency

$$\omega_{LF} = |p_i\omega_0 - q_i\omega(t)|, \quad (11)$$

can be obtained as

$$\begin{aligned} &\dot{\Phi}(t; q_i, p_i) + q_i\ H(P) \sum_{r_i, s_i} K(r_i, s_i) \\ &\times \sin(\frac{s_i}{q_i}\ \Phi(t; q_i, p_i) - \frac{\omega_0 t}{q_i}(q_i r_i - p_i s_i) + \Phi_0(r_i, s_i)) = \omega_{\mathrm{LF}}(p_i, q_i). \end{aligned} \quad (12)$$

The notation $K(r_i, s_i)$ means the effective gain at the harmonic r_i/s_i, $\Phi_0(r_i, s_i)$ is the reference angle and $H(P)$, where the operator $P = \frac{d}{dt}$, is the open loop transfer function. Solving (12) is a difficult task. Let us observe that the RF signal at frequency ω_0 acts as a periodic perturbation of the Adler's model of the PLL. If one neglects harmonic interactions, (12) may be simplified to the standard Arnold map model

$$\Phi_{n+1} = \Phi_n + 2\pi\Omega - c\ \sin\Phi_n, \quad (13)$$

where $\Omega = \frac{\omega}{\omega_0}$ is the bare frequency ratio and $c = \frac{K}{\omega_0}$. Such a nonlinear map is studied by introducing the winding number $\nu = \lim_{n\to\infty}(\Phi_n - \Phi_0)/(2\pi n)$. The limit exists everywhere as long as $c < 1$, the curve ν versus ω is a devil's staircase with steps attached to rational values $\Omega = \frac{p_i}{q_i}$ and width increasing with the coupling coefficient c. The phase-locking zones may overlap if $c > 1$ leading to chaos from quasi-periodicity [9].

The Arnold map is also a relevant model of a short Josephson junction shunted by a strong resistance R and driven by a periodic current of frequency ω_0 and amplitude I_0. Steps are found at the driving voltages $V_r = RI_0 = r(\hbar\omega_0/2e)$, r a rational number. Fundamental resonances $r = n$, n integer, have been used to achieve a voltage standard of relative uncertainty 10^{-7}.

To appreciate the impact of harmonics on the coupling coefficient one may observe that each harmonic of denominator q_i creates the same noise contribution $\delta\omega_{LF} = q_i\delta\omega(t)$. They are $\phi(q_i)$ of them, where $\phi(q_i)$ is the Euler totient function, that is the number of integers less or equal to q_i and prime to it; the average coupling coefficient is thus expected to be $1/\phi(q_i)$. In [2] a more refined model is developed based on the properties of prime numbers. It is based on defining a coupling coefficient as $c^* = c\Lambda(n; q_i, p_i)$ with $\Lambda(n; q_i, p_i)$ a generalized Mangoldt function. Mangoldt function is defined as

$$\Lambda(n) = \Lambda(n; 1, 1) = \begin{cases} \ln b & \text{if } n = b^k,\ b \text{ a prime,} \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

The generalized Mangoldt function attached to the resonance $\frac{p_i}{q_i}$ adds the restriction that n should also be congruent to $p_i \mathrm{mod(q_i)}$. The important result of that analysis is to exhibit a fluctuating average coefficient as follows

$$c^*_{\mathrm{av}}/c = \frac{1}{t}\sum_{n=1}^{t} \Lambda(n; q_i, p_i) = \frac{1}{\phi(q_i)} + \epsilon(t), \tag{15}$$

with $\epsilon(t) = O(t^{-1/2}\ln^2(t))$ which is known to be a good estimate as long as $q_i < \sqrt{t}$ [1]. The average coupling coefficient shows the expected dependance on q_i. In addition there is an arithmetical noise $\epsilon(t)$ with a low frequency dependance of the power spectrum reminding $1/f$ noise. Although that stage of the theory is not the last word of the story, it is quite satisfactory that this approach, based on phase-locking of the full set of harmonics, is accounting for the main aspects of $1/f$ noise found in experiments.

3. Hyperbolic Phase-Locking

The concept of an automorphic function is natural generalization of that of a periodic function. Furthermore, an automorphic form is a generalisation of the exponential function $e(z) = \exp(2i\pi z)$.

The citation is from Iwaniec's book [10].

As shown in the previous section the homodyne detector behaves as a diophantine approximator of the frequency ratio ν of input oscillators. The approach is very satisfactory in explaining frequency-locking effects in the open loop, but the recourse to non linear differential equations is necessary for the case of the closed loop. Now we attempt to develop a pure arithmetical frame to account for the phase-locking effects as well. This is done by normalizing the beat frequency with respect to the low pass cut-off frequency as $y = \frac{\omega_{LF}}{\omega_c}$ instead of the reference frequency. This is suggested by the geometry of continued fraction expansions which resorts to new mathematical concepts such as Ford circles, the hyperbolic half-plane, Möbius transformations, hyperbolic Laplace equation...

The hyperbolic half-plane is defined as the set of complex numbers z such that $\Im z \geq 0$. It is a rich mathematical object first studied by the mathematician Henri Poincaré at the end of nineteenth century. The geometry of $\mathcal{H}$ is related to continued fraction expansions and it is also a natural frame to study the prime numbers. It will be used also for studying phase-locking effects. We put[1].

$$z = \nu + iy, \quad \nu = \frac{\omega}{\omega_0}, \; 0 < y = \frac{\omega_{LF}}{\omega_c} < 1. \tag{16}$$

It is clear that $y > 0$ is a condition which is imposed by counting measurements. The second condition $y < 1$ results from the low pass filtering and will reveal important in our introduction of Ford circles below. Let us start with the continued fraction expansion (7) which is rewritten as $\{a_0; a_1, \cdots a_i\} = a_0 + \frac{1}{\{a_1; a_2 \cdots a_i\}} = \frac{p_i}{q_i}$.

By induction $p_0 = a_0$, $q_0 = 1$, $\{a_0, a_1\} = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1} = \frac{p_1}{q_1}$, so that

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_1 & p_0 \\ q_1 & q_0 \end{bmatrix},$$

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_i & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_i & p_{i-1} \\ q_i & q_{i-1} \end{bmatrix}. \tag{17}$$

Taking determinants and using a propery of continued fractions one gets $p_i q_{i-1} - p_{i-1} q_i = (-1)^{i-1}$. This suggests to associate to the convergents $\frac{p_i}{q_i}$ the group of Möbius transformations

$$z \to z' = \gamma(z) = \frac{p_i z + p_{i-1}}{q_i z + q_{i-1}}, \quad p_i q_{i-1} - p_{i-1} q_i = 1. \tag{18}$$

A Möbius transform map a point $z \in \mathcal{H}$, with $\mathcal{H}$ the upper half-plane $\Im z > 0$ into a point $z' \in \mathcal{H}$. Möbius transforms form a group Γ called the modular group. It is a discontinuous group that is used to tesselate $\mathcal{H}$ by copies of a fundamental domain $\mathcal{F}$ under the group action. The fundamental domain of Γ (or modular surface) is defined as $\mathcal{F} = \{z \in \mathcal{H} : \; |z| \geq 1, \; |\nu| \leq \frac{1}{2}\}$, and the family of domains $\{\gamma(\mathcal{F}), \gamma \in \Gamma\}$ induces a tesselation of $\mathcal{H}$ [10].

We are also interested by the Ford circles [12]: they are defined by the images of the filtering line $z = \nu + i$ under all Möbius transformations (18). Rewriting (18) as $q_i z' - p_i = -1/(q_i z + q_{i-1})$ and inserting $z = \nu + i$ in that expression one immediately gets

$$|z' - (\frac{p_i}{q_i} + \frac{i}{2q_i^2})| = \frac{1}{2q_i^2}. \tag{19}$$

which are circles of radius $\frac{1}{2q_i^2}$ centered at $\frac{p_i}{q_i} + \frac{i}{2q_i^2}$. To each $\frac{p_i}{q_i}$ a Ford circle in the upper-half plane can be attached, which is tangent to the real axis at $\nu = \frac{p_i}{q_i}$. Ford

[1] The imaginary number i such that $i^2 = -1$ should not be confused with the index i in integers p_i, q_i and related integers.

circles never intersect: they are tangent to each other if and only if they belong to fractions which are adjacent in the Farey sequence $\frac{0}{1} < \cdots \frac{p_1}{q_1} < \frac{p_1+p_2}{q_1+q_2} < \frac{q_1}{q_2} \cdots < \frac{1}{1}$. Ford circles can be considered a complex plane view of continued fractions.

Ford circles are also related to the wavefronts of an hyperbolic noise model: the Laplace equation for $\mathcal{H}$. To see this one first observe that $\mathcal{H}$ carries a non-Euclidean metric $dz = (d\nu^2 + dy^2)^{1/2}$. This is easily shown by using $\Im\gamma(z) = \frac{y}{|q_i z + q_{i-1}|^2}$ and $\frac{d\gamma(z)}{dz} = \frac{1}{(q_i z + q_{i-1})^2}$, since $|\frac{d\gamma(z)}{dz}| = \frac{\Im\gamma(z)}{y}$ under all group actions.

The invariance of the metric is inherited by the non-Euclidean Laplacian

$$\Delta = y^2(\frac{\partial^2}{\partial\nu^2} + \frac{\partial^2}{\partial y^2}). \tag{20}$$

We will be concerned by eigenfunctions $\Psi_s(z)$ of (20) as our dynamical model of phase-locking. The complex parameter s has been introduced as a label for the eigenvalues. The relevant wave equation is

$$(\Delta + \lambda)\Psi_s(z) = 0. \tag{21}$$

The solutions of (21) are called automorphic forms.

They are several ways of finding eigenfunctions $\Psi_s(z)$ for a given eigenvalue λ. We will first consider elementary power law solutions, then we will show that there is a method to generate a lot of eigenvalues out of a fixed $\Psi_s(z)$ by shifting to $\Psi_s(\gamma(z))$, and still more by averaging over selected $\gamma(z)$ in Γ.

The simplest solutions of (21) are horizontal waves of eigenvalue $\lambda = s(s-1)$ in the form of power laws

$$\Psi_s(z) = \left\{ \begin{array}{l} y^s \\ y^{1-s} \end{array} \right. . \tag{22}$$

Any other wave which satisfies (21) can be searched by shifting z to $\gamma(z)$ so that y goes to $\Im(\gamma(z))$ with the result

$$\Psi_s(z) = (\Im\gamma(z))^s = \frac{y^s}{|q_i z + q_{i-1}|^{2s}}. \tag{23}$$

The wavefronts for that solution are obtained by assigning to the quantity $y/(q_i z + q_{i-1})$ some constant value c. As a result one gets the equation $(x + q_{i-1}/q_i)^2 + (y - 1/(2q_i^2 c^2))^2 = 1/(4c^4 q_i^4)$ of circles tangent to the real axis at $x = -q_{i-1}/q_i$ and of radius $1/(2c^2 q_i^2)$. At $c = 1$ such a circle is obtained from the Ford circle of index i by an horizontal slip from $\nu = p_i/q_i$ to q_{i-1}/q_i. It contracts ($c < 1$) or expands ($c > 1$) remaining tangent to the real axis.

The more general solution of (21) is obtained by summing solutions (23) over γ transformations in Γ taken over the right coset $\Gamma_\infty \setminus \Gamma$ where Γ_∞ is the subgroup of translations $z \to z + n$. One gets [11]

$$\Psi_s(z) = y^s(1 + \sum_i \frac{1}{|q_i z + q_{i-1}|^{2s}}), \tag{24}$$

where the summation index i means that the summation should be applied to all Farey fractions $\frac{p_i}{q_i}$.

The whole solution can be decomposed into three contributions, the horizontal wave y^s is incident onto the filtering line, the reflected wave $S(s)y^{1-s}$ is scattered with a scattering coefficient $S(s)$ of modulus 1, whereas the remaining part $T_s(y,\nu)$ is a complex superposition of waves depending on y and the harmonics of $\exp(2i\pi\nu)$, but goes to zero away from the real line. More precisely one obtains $S(s) = A(s)Z(s)$, $Z(s) = \frac{\zeta(2s-1)}{\zeta(2s)}$, $A(s) = \frac{\Gamma(1/2)\Gamma(s-1/2)}{\Gamma(s)}$, $\Gamma(s)$ is the Gamma function and $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann zeta function. There is much to

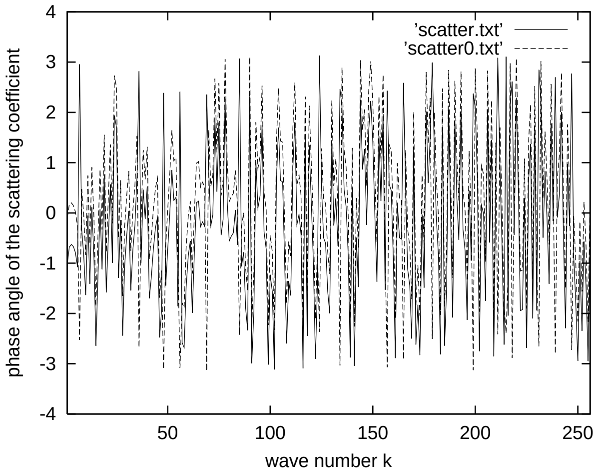

Fig 2. The phase angle $\kappa(k)$ for the scattering of noise waves on the modular surface. Plain lines: Exact phase factor. Dotted lines: Approximation based on the quotient $A(k)$ of two Riemann zeta functions at $2ik$ and $2ik+1$.

say about the singularities of the Riemann zeta function. It is analytic only on that part of complex plane where $\Re s > 1$. But in the scattering coefficient it appears in the extended Riemann zeta function $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ which is analytic over the whole complex plane. The scattering coefficient equals the quotient $S(s) = \xi(2s-1)/\xi(2s)$. One knows from the theory of zeta function that $s = \frac{1}{2}$ is an axis of symmetry for $\zeta(s)$, and according to Riemann hypothesis all non trivial zeros lie on that axis, with a very random distribution. An interesting case is thus to compute the scattering coefficient $S(s)$ along the critical line $s = \frac{1}{2} + ik$ in which case the superposition $T_s(y,\nu)$ also vanishes. As a result

$$S(k) \propto \exp(2i\kappa(k)) \quad \text{with } \kappa'(k) = \frac{d\ln Z(s)}{ds} \quad \text{at } s = \frac{1}{2} + ik. \tag{25}$$

In that expression we removed the smooth part $A(k)$ in the scattering coefficient so that its modulus is no longer 1 but its phase variability is still well described by the exponential factor above (see Fig. 2). The aim of this replacement is to get an explicit link of the counting function $\kappa'(k)$ to the Mangoldt function $\Lambda(n)$ already accounted in (14) as an effective coupling coefficient in the Arnold map (13). It is known that the logarithmic derivative of $\zeta(s)$ is $-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\geq 1} \frac{\Lambda(n)}{n^s}$. It is also easy to check that with $B(s) = \frac{\zeta(s)}{\zeta(s+1)}$, one gets $-\frac{B'(s)}{B(s)} = \sum_{n\geq 1} \frac{b(n)}{n^s}$ with

$b(n) = \frac{\Lambda(n)\phi(n)}{n}$ the modified Mangoldt function. Here $Z(s) = B(2s-1)$.

It is now clear that the scattering of waves in the hyperbolic plane has much to share with the phase-locking model formulated in Sect. 2. One argument in favor of that new frame is that the average modified Mangoldt function satisfies $B(t) = \frac{1}{t}\sum_{n\geq 1} b(n) = 1 + \epsilon_B(t)$, where $\epsilon_B(t)$ is an error term of power spectral density equal to $1/f^{2G}$ with $G \simeq 0.618$ the Golden ratio [12].

4. Quantum Phase-Locking

Apparently Dirac was the first to attempt a definition of a phase operator by means of an operator amplitude and phase decomposition. As we have discussed, with a complex c-number $a = Re^{i\Phi}$ one obtains the phase via $e^{i\Phi} = a/R$. Similarly, he sought to decompose the annihilation operator a into amplitude and phase components... After a brief calculation we obtain a relation indicating that the number operator N and phase operator Φ are canonically conjugate

$$[N, \Phi] = 1. \tag{26}$$

The equation immediately leads to a number-phase uncertainty relation which is often seen

$$\delta N \ \delta\Phi \geq 1/2. \tag{27}$$

However, all of the previous development founders upon closer examination.

This is taken from [13], a comprehensive review of the quantum phase problem. See also [14].

To approach the phase-locking problem within quantum mechanics one can start from the theory of the harmonic oscillator. The natural objects are the Fock states (the photon occupation states) $|n\rangle$ who live in an infinite dimensional Hilbert space. They are orthogonal to each other: $\langle n|m\rangle = \delta_{mn}$, where δ_{mn} is the Dirac symbol. The states form a complete set: $\sum_{n=0}^{\infty} |n\rangle\langle n| = 1$.

The annihilation operator removes one photon from the electromagnetic field

$$a|n\rangle = \sqrt{n}|n-1\rangle, n = 1, 2, \cdots \tag{28}$$

Similarly the creation operator $a^\dagger$ adds one photon: $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, n = 0, 1, \cdots$ There is the commutation relation $[a, a^\dagger] = 1$. The operator $N = aa^\dagger$ has the meaning of the particle number operator and satisfies the eigenvalue equation $N|n\rangle = n|n\rangle$.

Eigenvalues of the annihilation operator are the so-called coherent states $|\alpha\rangle$ [15]

$$a|\alpha\rangle = \alpha|\alpha\rangle, \quad \text{width } |\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n|n\rangle}{(n!)^{1/2}}, \tag{29}$$

where the eigenvalue α is a complex parameter which quantifies the intensity of the field. A single mode laser operated well above threshold generates a coherent state excitation. The electric field variation of a coherent state approaches that of a classical wave of stable amplitude and fixed phase. In the coordinate representation it is a minimal uncertainty state.

States of well defined phase escaping the inconsistencies of Dirac's formulation were build by Susskind and Glogower [16]. They correspond to the eigenvalues of the exponential operator

$$E = e^{i\Phi} = (N+1)^{-1/2} a = \sum_{n=0}^{\infty} |n\rangle\langle n+1|. \tag{30}$$

Using the Hermitian conjugate operator $E^\dagger = e^{-i\Psi}$, one gets $EE^\dagger = 1$, $E^\dagger E = 1 - |0\rangle\langle 0|$, i.e. the unitarity of E is spoiled by the vacuum-state projector $|0\rangle\langle 0|$. The Susskind-Glogower phase states satisfy the eigenvalue equation $E|\Psi\rangle = e^{i\psi}|\Psi\rangle$; they are given as

$$|\Psi\rangle = \sum_{n=0}^{\infty} e^{in\psi}|n\rangle. \tag{31}$$

Like the coherent states the phase states are non orthogonal and they form an overcomplete basis which solves the identity operator: $\frac{1}{2\pi}\int_{-\pi}^{\pi} d\psi |E\rangle\langle E| = 1$. The operator $\cos\Phi = \frac{1}{2}(E + E^\dagger)$ is used in the theory of Cooper pair box with a very thin junction when the junction energy $E_J \cos\Phi$ is higher than the electrostatic energy [18].

Further progress in the definition of phase operator was obtained by Pegg and Barnett. The phase states are now defined from the discrete Fourier transform (or more precisely the quantum Fourier transform since the superposition is on Fock states not on real numbers)

$$|\theta_p\rangle = \frac{1}{\sqrt{q}} \sum_{n=0}^{q-1} \exp(2i\pi \frac{p}{q} n)|n\rangle. \tag{32}$$

The states are eigenstates of the Hermitian phase operator

$$\Theta_q = \sum_{p=0}^{q-1} \theta_p |\theta_p\rangle\langle\theta_p|, \tag{33}$$

with $\theta_p = \theta_0 + 2\pi p/q$ and θ_0 is a reference angle. It is implicit in the definition (32) that the Hilbert space is of finite dimension q. The states $|\theta_p\rangle$ form an orthonormal set and in addition the projector over the subspace of phase states is $\sum_{p=0}^{q-1} |\theta_p\rangle\langle\theta_p| = 1_q$, where 1_q is the unitary operator. Given a state $|F\rangle$ one can write a probability distribution $|\langle\theta_p|F\rangle|^2$ which may be used to compute various moments, e.g. expectation values, variances. The key element of the formalism is that first the calculations are done in the subspace of dimension q, then the limit $q \to \infty$ is taken [17].

We are now in position to define a quantum phase-locking operator. Our viewpoint has much to share with the classical phase-locking problem as soon as one reinterpret the fraction $\frac{p}{q}$ in (32) as arising from the resonant interaction between two oscillators and the dimension q as a number which defines the resolution of the experiment. From now we emphasize such phase states $|\theta'_p\rangle$ which satisfy phase-locking properties and we impose the coprimality condition

$$(p, q) = 1. \tag{34}$$

The quantum phase-locking operator is defined as

$$\Theta_q^{\rm lock} = \sum_p \theta_p |\theta'_p\rangle\langle\theta'_p|, \tag{35}$$

with $\theta_p = 2\pi\frac{p}{q}$ and the notation p means summation from 0 to $q-1$ with $(p,q)=1$. Using (32) and (34) in (35) one obtains

$$\Theta_q^{\rm lock} = \frac{1}{q}\sum_{n,l} c_q(n-l)|n\rangle\langle l|, \tag{36}$$

where the range of values of n,l is from 0 to $\phi(q)$, and $\phi(q)$ is the Euler totient function. The coefficients in front of the outer products $|n\rangle\langle l|$ are the so-called Ramanujan sums

$$c_q(n) = \sum_p \exp(2i\pi\frac{p}{q}n) = \frac{\mu(q_1)\phi(q)}{\phi(q_1)}, \quad \text{with } \mathrm{q_1 = q/(q,n)}. \tag{37}$$

In the above equation $\mu(q)$ is the Möbius function, which is 0 if the prime number decomposition of q contains a square, 1 if $q=1$ and $(-1)^k$ if q is the product of k distinct primes. Ramanujan sums are relative integers which are quasi-periodic versus n with quasi-period $\phi(q)$, and aperiodic versus q with a type of variability imposed by the Möbius function. Ramanujan sums have been used for signal processing of low frequency noise [19]. With the Ramanujan sum expansion the modified Mangoldt function introduced at the end of Sect. 2 is the dual of Möbius function

$$b(n) = \frac{\phi(n)}{n}\Lambda(n) = \sum_{q\geq 1}\frac{\mu(q)}{\phi(q)}c_q(n) \tag{38}$$

This illustrates that many "interesting" arithmetical functions carry the structure of prime numbers. We already mentioned the relation $d\ln\zeta(s)/ds = \sum_{n\geq 1}\frac{\Lambda(n)}{n^s}$, but there is also the relation $1/\zeta(s) = \sum_{n\geq 1}\frac{\mu(n)}{n^s}$. There is a well known formulation of Riemann hypothesis from the summatory Möbius function $\sum_{n=1}^t \mu(n) = O(t^{1/2+\epsilon})$, whatever ϵ [1].

Given a state β one can calculate the expectation value of the quantum phase-locking operator as

$$\langle\Theta_q^{\rm lock}\rangle = \sum_p \theta_p \langle\theta'_p|\beta\rangle^2. \tag{39}$$

If one uses the finite form of Susskind-Glogower phase states (31) and a real parameter β

$$|\beta\rangle = \frac{1}{\sqrt{q}}\sum_{n=0}^{q-1}\exp(in\beta)|n\rangle, \tag{40}$$

the expectation value of the locked phase becomes

$$\langle\Theta_q^{\rm lock}\rangle = \frac{\pi}{q^2}\sum_{n,l} c_q(l-n)\exp(i\beta(n-l)). \tag{41}$$

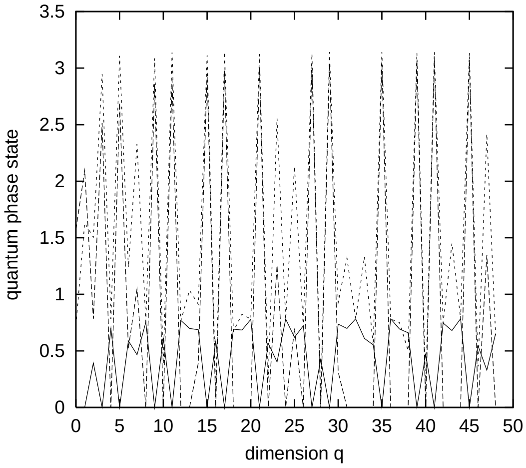

Fig 3. Oscillations in the expectation value (41) of the locked phase at $\beta = 1$ (dotted lines) and their squeezing at $\beta = 0$ (plain lines). The brokenhearted line which touches the horizontal axis is $\pi\Lambda(q)/\ln q$

For $\beta = 1$ it is found that $\langle\Theta_q^{\mathrm{lock}}\rangle$ has the more pronounced peaks are at such values of q which are powers of a prime number. It can be approximated by the normalized Mangoldt function $\pi\Lambda(q)/\ln q$ as shown on Fig. 3. For $\beta = 0$ the expectation value of $\langle\Theta_q^{\mathrm{lock}}\rangle$ is much lower. The parameter β can be used to minimize the phase uncertainty well below the classical value [3].

A remarkable model of the quantum phase-locking effect and its relation to prime number theory has been constructed by Bost and Connes [20]. Instead of an ad-hoc quantum phase operator as ([17]) or (36), it is based on the formulation of a dynamical system and its associated quantum statistics. The dynamical system is first defined by an Hamiltonian operator H_0 with eigenvalues equal to the logarithms of integers

$$H_0|n\rangle = \ln n|n\rangle. \tag{42}$$

Using the relations $\exp(-\tilde{\beta}H_0)|n\rangle = \exp(-\tilde{\beta}\ln n)|n\rangle = n^{-\tilde{\beta}}|n\rangle$, it follows that the partition function of the model at the inverse temperature $\tilde{\beta}$ is

$$\mathrm{Trace}(\exp(-\beta\tilde{\mathrm{H}}_0)) = \sum_{\mathrm{n}=1}^{\infty}\mathrm{n}^{-\tilde{\beta}} = \zeta(\tilde{\beta}), \tag{43}$$

where $\zeta(\tilde{\beta})$ is the Riemann zeta function already met in Sec. 3 in the definition of the hyperbolic scattering coefficient (25). In quantum statistical mechanics, given an observable Hermitian operator M one has the Hamiltonian evolution $\sigma_t(M)$ versus time t

$$\sigma_t(M) = e^{itH_0}Me^{-itH_0}, \tag{44}$$

and the expectation value of M is the Gibbs state $\mathrm{Gibbs(M)} = \mathrm{Trace}(\mathrm{M}\exp(-\tilde{\beta}\mathrm{H}_0))/\mathrm{Trace}(\exp(-\tilde{\beta}\mathrm{H}_0)$. In Bost and Connes ap-

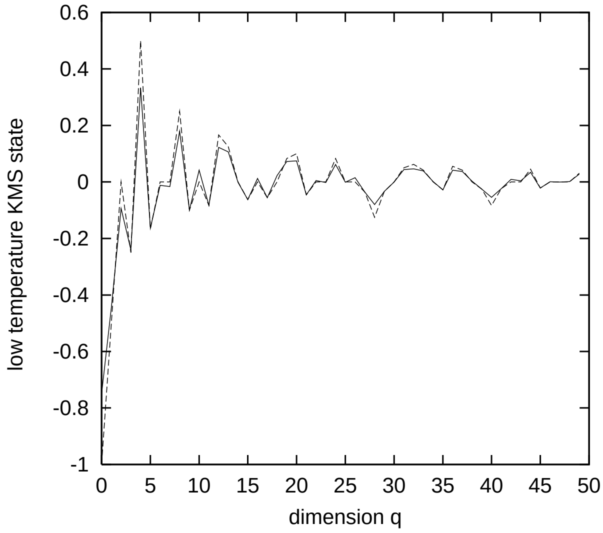

Fig 4. Phase expectation value (47) in Bost and Connes model at the inverse temperature $\beta = 3$ (plain lines) in comparison to the function $\mu(q)/\phi(q)$ (dotted lines)

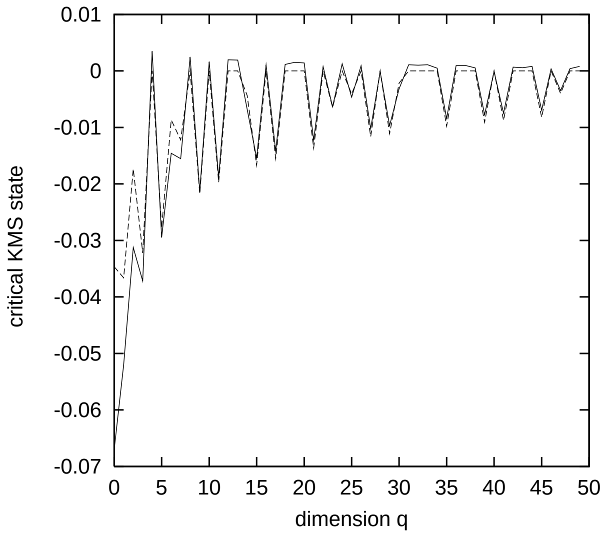

Fig 5. Phase expectation value (47) in Bost and Connes model at the inverse temperature $\beta = 1+\epsilon$, $\epsilon = 0.1$ (plain lines) in comparison to the function $-\Lambda(q)\epsilon/q$ (dotted lines).

proach the observables belong to an algebra of operators μ_q and $e_q^{(p)}$ which are defined by their action on the occupation numbers $|n\rangle$ as

$$\mu_q|n\rangle = |qn\rangle, \tag{45}$$

$$e_q^{(p)}|n\rangle = \exp(\tfrac{2i\pi pn}{q})|n\rangle. \tag{46}$$

The first operator μ_q acts as a shift in the space of number states; the second one is such that its action encodes the individuals in the quantum Fourier transform (32). Like in the quantum phase-locking operator one uses the coprimality condition (34) to distinguish in (46) the primitive roots of unity $\exp(2i\pi p/q)$, $(p,q)=1$. One can show that there is a hidden symmetry group which is used to label the elements of the algebra[2]. Using the action of the group, the Gibbs state is replaced by the so-called Kubo-Martin-Schwinger (or KMS) state. The system exhibits a phase transition with spontaneous symmetry breaking at the inverse temperature $\tilde{\beta}=1$ which corresponds to the unique pole of the Riemann zeta function $\zeta(\tilde{\beta})$. At low temperature $\tilde{\beta}>1$ one gets, after tricky calculations, the expectation value of the phase operator which replaces (41) in the following form [20]

$$\mathrm{KMS}(\mathrm{e}_\mathrm{q}^{(\mathrm{p})}) = \mathrm{q}^{-\tilde{\beta}} \prod_{\substack{\mathrm{p\ divides\ q} \\ \mathrm{p\ prime}}} \frac{1-\mathrm{p}^{\tilde{\beta}-1}}{1-\mathrm{p}^{-1}}. \tag{47}$$

The KMS state is represented for two limiting cases, the low temperature limit $\beta \gg 1$ (Fig. 4) and the critical case $\beta = 1+\epsilon$ (Fig. 5), with $\epsilon \simeq 0$. In these limits one has respectively $\mathrm{KMS}_{\beta\gg 1}(\mathrm{q}) = \frac{\mu(\mathrm{q})}{\phi(\mathrm{q})}$ and $\mathrm{KMS}_{1+\epsilon} \simeq \frac{-\Lambda(\mathrm{q})\epsilon}{\mathrm{q}}$. In the low temperature limit the spectrum (38) corresponding to the Ramanujan sum expansion of the modified Mangoldt function $b(n)$ is recovered. Close to the critical point $\beta = 1+\epsilon$ the oscillations are proportional to $\Lambda(q) \simeq b(q)$ and are of very small amplitude due to the squeezing coefficient ϵ. A comparable squeezing effect was already observed in the expectation value (39) of the quantum phase operator (see Fig. 3). After the phenomenological model (15) and the hyperbolic model (25), the Bost and Connes cyclotomic model also points to the Mangoldt function as a source of low frequency fluctuations. In the last case the model is associated to the spontaneous symmetry breaking and the squeezing of phase oscillations at the critical KMS state.

As a conclusion phase-locking evokes Einstein "spooky" action at a distance. It is a very important concept in phase sensitive communication circuits. Quantum phase-locking may become synonymous of phase entanglement, be used to such tasks as remote synchronization, quantum noise reduction and as a quantum resource for the circuits of quantum communications.

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[2]This is the Galois group $W = Gal(\mathcal{Q}^{\mathrm{cycl}}/\mathcal{Q})$ of the cyclotomic extension on the field of rational numbers $\mathcal{Q}$.

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