THE IMPACT OF PRIME NUMBER THEORY ON FREQUENCY METROLOGY

M. PLANAT

Laboratoire de Physique et Métrologie des Oscillateurs du CNRS
32 Avenue de l'Observatoire, 25044 Besanćon Cedex, France
E-mail: planat@lpmo.edu

1 Arithmetic of Beat Frequency Measurements

Low frequency noise of electronic oscillators is usefully interpreted in terms of arithmetic[1]: this is because the measurement of the frequency $f(t)$ of an oscillator under test compares the one f_0 of a reference oscillator thanks to a nonlinear mixing set-up and a filter. The beat frequency

$$f_B = |p_i f_0 - q_i f(t)|\,, \quad \text{with } \mathrm{p_i} \text{ and } \mathrm{q_i} \text{ integers,} \tag{1}$$

follows from the continued fraction expansion of the frequency ratio $\nu = \frac{f}{f_0} = [a_0; a_1, a_2, ...a_i, a, ...] = a_0 + 1/\{a_1 + 1/\{a_2 + ... + 1/\{a_i + 1/\{a...\}\}\}\} = \frac{p_i(a)}{q_i(a)} \simeq \frac{p_i}{q_i}$ of the input oscillators. Here $a_{\min} \leq a \leq a_{\max}$, with $a_{\min} = \lfloor \frac{f_0}{f_c q_i} \rfloor$, $a_{\max} = \lfloor \frac{f_0}{f_d q_i} \rfloor$ and f_c and f_d are the low and high frequency cut-off of the filter. Since $a \gg 1$ in typical measurements, the beat note is well approximated by the convergent (1) which restricts to the partial quotient a_i in the expansion. Fig. 1 shows a schematic of the resulting intermodulation spectrum [2].

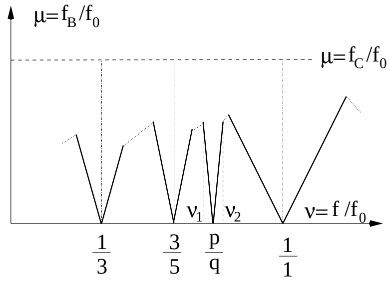

Figure 1. The intermodulation spectrum at the output of the mixer+filter set-up.

In some bad circumstances [1] the partial quotients after a don't play any role and frequency jumps occurs randomly at definite values of a leading to a large white frequency noise arising from the detection set-up instead of the oscillator under test. This can be compared to the measurement of time from a moon-sun calendar. Early calendars have been devised from the motion of moon and sun as observed from the earth. The continued fraction expansion of the ratio ν between the sun year and the moon year is

$$\nu = \frac{365.242191}{29.530589} = [12; 2, 1, 2, 1, 1, 17, \ldots] \tag{2}$$

The first approximation $\nu = 12$ (with 354 days) can be corrected by adding one month every two years, the second one (with 369 days) may be corrected by adding one month every three years and so on. Fluctuation of the integer a in the frequency measurement set-up has the same aim to correct the measurement versus time.

2 Arithmetic of Phase Locking and $1/f$ Frequency Noise

If one can track the frequency $f(t)$ by that f_0 of a voltage controlled oscillator one gets a phase locked loop[3]. To a first approximation the phase locking model [2] for the phase shift θ_n between oscillators is the Arnold map

$$\theta_{n+1} = \theta_n + 2\pi\Omega - c\ \sin\theta_n, \tag{3}$$

where $\Omega = \frac{f}{f_0}$ is the bare frequency ratio, $c = \frac{K}{f_0}$ and K is the open loop gain. Such a non linear map is studied by introducing the winding number $\nu = \lim_{n\to\infty}(\theta_n - \theta_0)/(2\pi n)$. The limit exists everywhere as long as $c < 1$, the curve ν versus Ω is a devil's staircase with steps attached to rational values of $\Omega = \frac{p_i}{q_i}$ and with width increasing with the coupling coefficient c. The phase locking zones may overlap if $c > 1$ leading to chaos from quasiperiodicity [4].

To appreciate the impact of harmonics on the coupling coefficient one can observe that each harmonic of denominator q_i leads to the same fluctuating frequency $\delta f_B = q_i \delta f(t)$. There are $\phi(q_i)$ of them, where $\phi(q_i)$ is Euler totient function, that is the number of integers less or equal to q_i and prime to it; the average coupling coefficient is thus expected to be $1/\phi(q_i)$[a]. We developed a more refined model based on the properties of primes by defining the coupling coefficient as $c^* = c\Lambda(n; q_i, p_i)$ with

$$\Lambda(n; q_i, p_i) = \begin{cases} \ln b & \text{if } n = b^k,\ b \text{ a prime and } \mathrm{n \equiv p_i \ mod(q_i)} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

[a] This should be corrected to account for the symmetry of the mixer. In the balanced phase bridge we used in the experiments, only harmonics with odd values of p_i and q_i contribute significantly.

This means a non zero coupling to harmonics at times $n = p_i + q_i\ l$, l integer whenever n is a power of a prime; the coupling at the fundamental mode is the so-called von Mangoldt function $\Lambda(n)$[5].

According to the generalized Riemann hypothesis one gets the average[6]

$$c^*_{\mathrm{av}}/c = \frac{1}{t}\sum_{n=1}^{t}\Lambda(n; q_i, p_i) = \frac{1}{\phi(q_i)} + \epsilon(t), \tag{5}$$

with $\epsilon(t) = 0(t^{-1/2} ln^2(t))$ which is a good approximation as long as $q_i < \sqrt{t}$. A better estimate may also be obtained at larger q_i[6].

For $\frac{p_i}{q_i} = \frac{1}{1}$ the fluctuating term may be expressed in terms of the zeros of the Riemann zeta function $\zeta(s)$. The trivial ones at $s = -2l$, l integer, connect to Bernouilli polynomials; according to Riemann hypothesis (still unproved), they are infinitely many non trivial zeros at the critical line $s = \frac{1}{2}$ which are randomly distributed. In the general case of the $\frac{p_i}{q_i}$ harmonic, $\zeta(s)$ generalizes to a Dirichlet series. One finds numerically that the power spectral density of the fluctuating term looks like a $1/f$ noise [1].

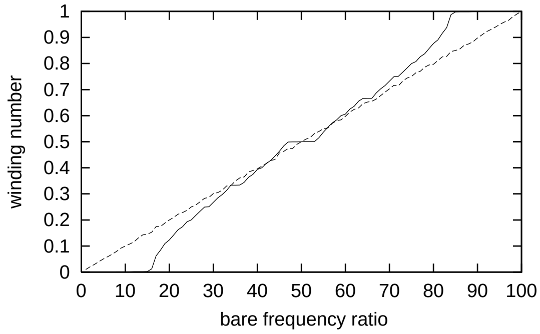

Figure 2. Phase locking steps for the Arnold map (plain line), and lack of phase locking steps for the Arnold–von Mangoldt map (dotted line); coupling coefficient $c = 1$.

We have studied numerically phase locking properties of mapping (3) with c^* as the coupling coefficient in place of the standard one c. The more drastic effect is to prevent the phase locking of the oscillators. Fig. 2 shows that the phase locking steps are removed. In addition the $1/f$ noise in the power spectral density of $\epsilon(t)$ converts to $1/f$ noise in the fluctuation of the frequency ratio ν as shown in Fig. 3.

In conclusion the present theory connects prime numbers to the metrology of time and $1/f$ frequency noise in the context

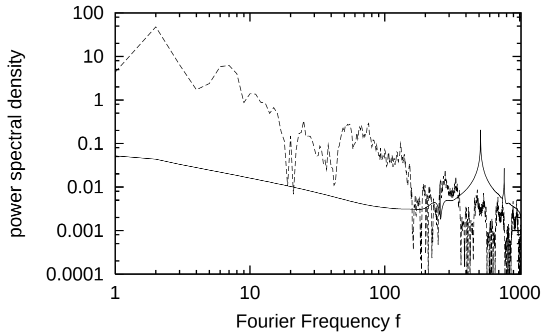

Figure 3. $1/f$ noise of the winding number in the Arnold–von Mangoldt map (dotted line) in comparison to the Arnold map (plain line); coupling coefficient $c = 0.2$.

of phase locking of oscillators. For others links between number theory and physics see http://www.ift.uni.wroc.pl/~mwolf and http://www.maths.ex.ac.uk/~mwatkins/zeta/physics.htm.

3 Annex 1: The $1/f$ Noise Term and Riemann Hypothesis

We remind the relationship between the fluctuating term $\epsilon(t)$ and the theory of the Riemann zeta function[5].

Riemann zeta function $\zeta(s)$ is defined from the Euler product

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{b \text{ prime}} \frac{1}{1 - \frac{1}{b^s}} \quad \text{where } \Re(s) > 1. \tag{6}$$

Riemann's great achievment in 1859 was his ability to complete the formula to the whole complex plane of the parameter s. By logarithmic derivation (6) can be rewritten as

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= \sum_{n=1}^{\infty} \Lambda(n) n^{-s} \\ &= \int_1^{\infty} t^{-s} d\psi(t) = s \int_1^{\infty} t^{-s-1} \psi(t) dt \quad \text{with } \Re(\mathrm{s}) > 1, \end{aligned} \tag{7}$$

with the von Mangoldt function $\Lambda(n) = \ln(b)$ if $n = b^k$, b a prime and 0 otherwise. Function $\psi(t) = \sum_{n \leq t} \Lambda(n)$ is the summatory function.

The inverse transform

$$\psi(t) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} -\frac{\zeta'(s)}{\zeta(s)} t^s \frac{ds}{s} \quad \text{with c} = \Re(\text{s}) > 1, \tag{8}$$

allows an estimate of $\psi(t)$ if one knows the singularities of $\zeta(s)$. The pole of $-\frac{\zeta'(s)}{\zeta(s)}$ at $s = 1$ contributes t; the pole $1/s$ at $s = 0$ contributes $-\frac{\zeta'(0)}{\zeta(0)} = -\ln(2\pi)$ and the zeros ρ contribute $-\frac{t^\rho}{\rho}$. One gets

$$\begin{aligned} &\psi(t) = t(1 + \epsilon(t)) \\ &\text{with } \mathrm{t}\epsilon(\mathrm{t}) = -\ln(2\pi) - \frac{1}{2}\ln(1 - \mathrm{t}^{-2}) - \sum_\rho \frac{\mathrm{t}^\rho}{\rho}. \end{aligned} \tag{9}$$

The second term in $\epsilon(t)$ is due to the trivial zeros of $\zeta(s)$ which are located at $s = -2l$ (l a positive integer). The third term is due to the remaining zeros of $\zeta(s)$. Billions of them have been computed; all are found to be located on the line $s = \frac{1}{2}$. Riemann hypothesis is the (unsolved) conjecture that all non trivial zeros belong to the critical line. These zeros are very irregularly spaced and are responsible for the very irregular shape of the error term as shown in Fig. 4. It was shown numerically[1] that the power spectral density of $\epsilon(t)$ has

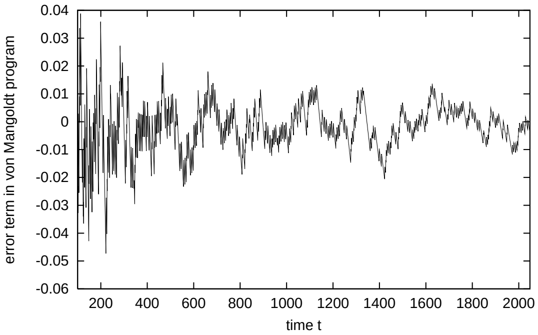

Figure 4. The error term $\epsilon(t)$ in the summatory function $\psi(t)$.

a $1/f$ dependance on the Fourier frequency f.

The fluctuating term $\sum_\rho \frac{t^\rho}{\rho}$, where $\rho = \frac{1}{2} + iy$, can be bounded if one knows the number $N(y)$ of zeros between 0 and y. Since $N(y) < y\ln(y)$, assuming Riemann hypothesis, this implies the von Koch estimate $\epsilon(t) = O(t^{-1/2}\ln^2(t))$ of the error term[6].

4 Annex 2: Harmonic Interactions, $1/f$ Noise and the Generalized Riemann Hypothesis

Euler's identity (6) can be generalized to the Dirichlet L-series

$$L(s,\kappa) = \sum_{n=1}^{\infty} \frac{\kappa(n)}{n^s} = \prod_{b \text{ prime}} \frac{1}{1-\kappa(b)b^{-s}} \quad \Re(\mathrm{s}) > 1$$
$$\text{with } \kappa(\mathrm{n}) = \kappa(\mathrm{n}) \bmod(\mathrm{q}) \ \text{ for } (\mathrm{n},\mathrm{q}) = 1 \ \text{ and } 0 \ \text{ otherwise.} \tag{10}$$

In (10) the notation $(n, q) = 1$ means that n and q are coprimes[b]. The Dirichlet character $\kappa(n)$ is thus a multiplicative function.

Using $\Lambda(n; q, p)$ in place of $\Lambda(n)$, (7)-(9) can be generalized to the summatory function $\psi(n; q, p) = \sum_{n \le t} \Lambda(n; q, p)$, where $n \equiv p$ mod(q), with the result

$$\psi(t; q, p) = t(1 + \epsilon(t; q, p))$$
$$\text{with } \mathrm{t}\epsilon(\mathrm{t};\mathrm{q},\mathrm{p}) = -\frac{\mathrm{L}'(0,\kappa)}{\mathrm{L}(0,\kappa)} + \sum_{\mathrm{m}\ge 1} \frac{\mathrm{t}^{1-2\mathrm{m}}}{2\mathrm{m}-1} - \sum_{\rho} \frac{\mathrm{t}^{\rho}}{\rho}. \tag{11}$$

The error term is shown in Fig. 5. As above its power spectral density approximates a $1/f$ law. The fluctuating term can be bounded assuming

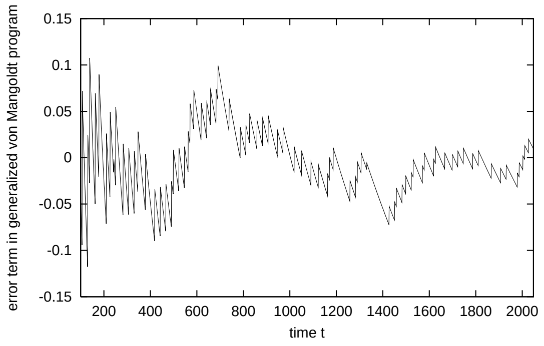

Figure 5. The error term $\epsilon(t; q, p)$ in the summatory function $\psi(t; q, p)$; $p/q = 3/8$.

the generalized Riemann hypothesis that all non trivial zeros of L-functions belongs to the critical line $s = \frac{1}{2}$. The same (poor) estimate[6] $\epsilon(t; q, p) = O(t^{-1/2}) \ln^2(t))$ follows.

[b] In this section we use the notation p, q instead of p_i, q_i

5 Annex3: $1/f$ Frequency Noise and the Phase Locked Loop

From numerous experiments we found (see Fig. 6) that $1/f$ frequency noise is a property of an (unlocked) phase locked loop (PLL)[3]. It is thus of great interest to relate error terms of prime number theory to the study of the phase locked loop.

If one accounts for the whole set of harmonics, the differential equation for the phase shift $\theta(t; q, p)$ at the harmonic (p, q) can be written[2]

$$\dot{\theta}(t; q, p) + q\ H(P) \sum_{r,s} K(r, s) \sin(\frac{s}{q}\ \theta(t; q, p) - \frac{\omega_0 t}{q}(qr - ps) + \theta_0(r, s)) \\ = \omega_B(p, q), \quad (12)$$

where $\omega_B = 2\pi f_B$, f_B is the beat frequency as given in (1), $K(r, s)$ is the effective gain at harmonic (r, s) and $H(P)$, where $P = \frac{d}{dt}$, is the open loop transfer function. Solving (12) is formidable task. It is enough here to observe that the reference signal at frequency $f_0 = 2\pi/\omega_0$ acts as a periodic perturbation of the standard model of the PLL.

If one neglects harmonic interactions, (12) simplifies to the standard Arnold map model[2] (3). Phase locking zones corresponds to the stairs of the curve $\nu = \lim_{n\to\infty}(\theta_n - \theta_0)/(2\pi n)$ versus Ω, as it is explained in Sect. 2. The open loop gain K in the PLL reflects into the coupling coefficient

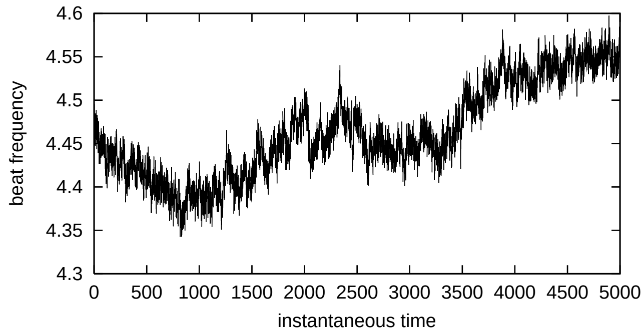

Figure 6. Beat frequency (in Hz) close to the phase locked zone 1/1 of a PLL (reference frequency: $f_0 = 5\ MHz$). The power spectrum of these records has a pure $1/f$ dependance.

$c = K/f_0 \ll 1$. In such a case either the loop is phase locked at some harmonic $\nu = \frac{p}{q}$ or ν is an irrational number. Quasiperiodic chaos may occur only at $c > 1$. Accounting for the effects of harmonics we can expect that the coupling coefficient generalizes as $c^* = c(1 + \alpha/\phi(q))$, α a constant, since each harmonic of denominator q has the same effect $\delta f_B = q\delta f(t)$ on the beat frequency and they are $\phi(q)$ of them, with $\phi(q)$ the Euler function. This leads to a lack of phase locking as in the uncoupled case $c = 0$; but no chaos is produced as it is the case for the linear map.

We are thus led to the unorthodox conclusion that the effect of harmonics may be to produce a digital modulation of the coupling coefficient as $c_{\mathrm{av}} = c(1 + \alpha\Omega(n; q, p))$. To a first approximation the average coupling coefficient still is c^*, but there is the error term $\epsilon(t; q, p)$ as given in (5). The $1/f$ noise which is present in c^*_{av} moves to the winding number ν as it is shown in Fig. 3. This would explain that 1/F FREQUENCY NOISE IS A UNIVERSAL PROPERTY OF UNLOCKED PHASE LOCKED LOOPS as found from experiments.

Acknowledgments

I thanks my summer students J.P. Marillet and E. Henry for their help in the experiments leading to the present theory.

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