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0.1 Universality Principle and Riemann hypothesis

The basic definition of $\zeta(s = x + iy)$ based on the product formula does not converge for $Re[s] \leq 1$. One can however define 'universal' ζ, call it $\hat{\zeta}$, as the product of the partition functions $Z_{p_1}(s) = 1/(1-p^{-s})$, in the subset of complex plane, where the factors Z_{p_i} are complex algebraic numbers. The idea is to regard the value of $\hat{\zeta}$ as an element of an infinite-dimensional algebraic extension of the rationals containing all roots of primes. $\hat{\zeta}$ can be regarded as a vector with infinite number of components and is completely well defined despite the fact that the product expansion does not converge as an ordinary complex number unless one somehow specifies how the 'producting' is done.

In case that the factors $|Z_{p_1}|^2$ of the partition functions $Z_{p_1} = 1/(1-p^{-z})$ are complex rationals, one can rewrite the product formula by applying adelic formula to the norm squared $|Z_{p_1}|^2$ appearing in the product formula. The basic hypothesis is that the product of the p-adic norms of the complex norm squared of the function $\hat{\zeta}$ defined by the product formula obtained by changing the order of producting gives the norm squared of the analytically continued ζ in the region ($Re[s] < 1$, $Im[s] \neq 0$) at the points, where the factors $|Z_{p_1}|^2$ are algebraic numbers: $|\hat{\zeta}|^2 = \prod_p N_p(|\hat{\zeta}|^2) = |\zeta|^2$. A milder version of this hypothesis is that the product of the p-adic norms squared of $|\hat{\zeta}|^2$ converges to some function proportional to $|\zeta|^2$.

If this hypothesis is correct, the following vision giving good hopes about the proof of the Riemann hypothesis, suggests itself.

a) $|\hat{\zeta}|^2$ is a number in an infinite-dimensional algebraic extension of rationals and can vanish only if it contains a rational factor which vanishes. The vanishing of this factor is possible if it is a product of an infinite number of moduli squared $|Z_{p_1}(z)|^2$ having a rational value. For the values of y for which this is true on the line $Re[s] = n + 1/2$ correspond to the phases p_1^{-iy} having the following general form.

$$
\begin{aligned}
&p^{-iy} = U_1 U = \frac{(r_1 + is_1\sqrt{k(p_1,y)})}{\sqrt{p_1}} \times \frac{(r + is\sqrt{k(p_1,y)})}{n_1} \quad ,\\
&r_1^2 + s_1^2 k(p_1,y) = p_1 \quad ,\\
&r^2 + s^2 k(p_1,y) = n_1^2 \quad .
\end{aligned}
$$

$r_1^2 + s_1^2 k(p_1,y) = p_1$ condition is solved by $k(p_1,y) = \sqrt{p_1 - m^2}$, $m < \sqrt{p}$. $r^2 + s^2 k(p_1,y) = n_1^2$ condition is satisfied if U is a product of even powers of the phases of type U_1. Unless $k(p_1,y)$ is not square, the phases correspond to orthogonal triangles with one short side having integer valued length and the other sides having integer valued length squared.

b) If y defines rational value of $|Z_{p_1}(z)|^2$, also its integer multiples ny do the same. If the values of integers $k(p_1,y)$ do not depend on the value of y, the allowed values of y generate an additive group having integers as a coefficient ring. Even powers of the phases guaranteing the rationality of $|Z_{p_1}(z)|^2$ on the line $Re[s] = 1/2$, guarantee rationality on the lines $Re[s] = n$.

c) Especially important subset of these phases correspond to the choice $k_1 = 1$. These phases correspond to Gaussian primes having the form $G = r_1 + is_1$, $r_1^2 + s_1^2 = p_1$, $p_1 \ mod \ 4 = 1$, and can compensate the irrationality of the $p_1^{-n-1/2}$ factor only in this case. The products of the squares of

Gaussian primes define Pythagorean triangles and the corresponding phases are rational. Rather interestingly, the linear superpositions $y = n_1y_2 + n_2y_2$ of only *two* Pythagorean values of y_i form a dense subset of reals. Eisenstein primes having the general form r_1+s_1w, $w = -1/2\pm\sqrt{3/2}$, $r_1^2+s_1^2-r_1s_1 = p_1$, $p_1\ mod\ 3 = 1$, are second, probably very important class of complex primes. They can compensate the irrationality of the $p_1^{-n-1/2}$ factor for $p_1\ mod\ 3 = 1$ (note that the 1/2 is not relevant for the phase). Also other phases are needed since for primes satisfying $p_1\ mod\ 4 = 3$ and $p_1\ mod\ 3 = 2$ simultaneously neither Gaussian nor Eisenstein primes can compensate the irrationality of the $p_1^{-1/2}p_1^{-iy}$ factor.

d) The lines on which the real parts for an infinite number of factors Z_{p_1} can be rational, correspond to the lines $Re[s] = n/2$. This in turns leads to the conclusion that the norm squared of $\hat{\zeta}$ can vanish only on the lines $Re[s] = n/2$. If the norm squared of the $\hat{\zeta}$ coincides with the norm squared of the analytically continued ζ, Riemann hypothesis follows since it is known that the lines $Re[s] = n/2$, $n \neq 1$ do not contain zeros of ζ.

In the following this vision is developed in detail and it is shown that it survives the basic tests.

0.2 Detailed realization of the Universality Principle

Universality Principle states that ζ vanishes only if $|\hat{\zeta}|^2$ understood as a number in an infinite-dimensional algebraic extension of rationals vanishes and hence must contain a rational factor resulting from an infinite number of rational factors Z_{p_1}. This hypothesis alone makes Riemann hypothesis very plausible. In this section an attempt to reduce the Universality Principle to something more conrete is made. Adelic formula and the hypothesis that the norm of $|\hat{\zeta}|^2$ defined by the modified adelic formula equals to $|\zeta|^2$ are described and shown to imply Universality Principle if the modified adelic formula defines a norm in the infinite-dimensional algebraic extension of rationals. The conditions guaranteing the rationality and the reduction of the p-adic norm of $|Z_{p_1}|^2$ are derived, and the connection between Pythagorean phases and basic facts about Gaussian and Eisenstein primes are summarized.

0.2.1 Modified adelic formula and Universality Principle

Although the product representation of ζ does not converge absolutely for $Re[s] \leq 1$, one can consider the possibility that the convergence of the function $\hat{\zeta}$ defined by the product representation occurs in some exceptional points in some natural sense. The points at which the value of $\hat{\zeta}$ belongs to the infinite-dimensional algebraic extension of rationals are obviously excellent candidates for these points. $\hat{\zeta}$ identified as an element of this algebraic extension certainly exists mathematically as a vector with an infinite number of components. The convergence in the strong sense would mean that the interpretation of the algebraic numbers of the algebraic extension as real numbers in the expression of $\hat{\zeta}$ gives the analytically continued ζ somehow. In the weak sense the convergence would mean that the complex norm squared for $\hat{\zeta}$, if defined in a suitable sense, equals or is proportional, to the norm squared of the analytically continued ζ.

1. Modified Adelic formula and Universality Principle

The fact that the product formula for ζ at rational points converges only conditionally, suggests that one should be able to device a natural method of 'producting' giving rise to the norm squared of the analytically continued ζ. Adelic formula provides very attractive approach to this problem (the appearence of the norm squared instead of norm is motivated by the Adelic formula).

The adelic formula expresses the real norm of a rational number as a product of the inverses of the p-adic norms

$$\frac{1}{|x|_R} = \prod_p |x|_p \ . \tag{1}$$

This formula generalizes also to the norms of the complex rationals. How to generalize this formula to the infinite-dimensional algebraic extension of rationals? The simplest possibility is to write the complex norm squared as vector in the infinite-dimensional extension having rational coefficients and to apply adelic formula to each factor separately.

$$|x|_R = \sum_k e_R^{k)} \prod_p |\frac{1}{x_k}|_p \ ,$$

$$|x| \ = \ \sum_k e^{k)} x_k \ . \quad (2)$$

Here $e^{k)}$ denote the units of the infinite-dimensional algebraic extension (products of roots of primes and analogous to imaginary unit) and $e_R^{k)}$ denote the evaluations of these units identified as real numbers. The resulting norm is indeed equal to the real norm when the resulting number is interpreted as a real number.

In the case that the factors Z_{p_1} of ζ are complex rationals, one can write the real norm of the real ζ for $Re[s] > 1$ as a product

$$|\zeta(z)|^2 = \ = \ \prod_{p_1} \left[\prod_p N_p(|\frac{1}{Z_{p_1}(z)}|^2) \right] \equiv \prod_{p_1} \left[\prod_p N_p(|Z_{p_1}^{p)}(z)|^2) \right] \ . \quad (3)$$

Here $N_p(x)$ denotes the p-adic norm of number x. This formula explains why one must define the p-adic zeta as an arithmetic inverse of the real ζ. The generalization of this formula to the case that $\hat{\zeta}^2$ has values in the set of the complex rationals is straighforward.

The problem with this representation is that the product over primes p_1 does not converge in an absolute sense for $Re[s] \leq 1$. By a suitable rearrangment of a conditionally convergent product a convergence to any number can be achieved. This suggests that one could find some unique manner to rearrange the terms to a convergent expression converging to $|\zeta|^2$. A unique definition indeed suggests itself: the analytic continuation of ζ from the region $Re[s] > 1$ might be equivalent with the exchange of the order of 'producting' in the expression of ζ:

$$\begin{aligned} |\hat{\zeta}(z)|^2 \ &= \ \prod_p N_p(|\frac{1}{\zeta(z)}|^2 = \prod_p \left[\prod_{p_1} N_p(|\frac{1}{Z_{p_1}(z)}|) \right] \\ &= \ \prod_p N_p(|\frac{1}{\zeta}|^2) = \prod_p N_p(|\zeta^{p)}|)^2 \ . \end{aligned} \quad (4)$$

The minimal working hypothesis is that $|\hat{\zeta}|^2$ defined as the product its p-adic norms equals to $|\zeta|^2$ at points, where its values are *rational*:

$$\prod_p N_p(|\hat{\zeta}|^2) \ = \ |\zeta|^2 \ . \quad (5)$$

The generalization to the algebraic extension of rationals is straighforward since the p-adic norm squared is sum over the p-adic norms of the components of the algebraic extension with various units $e^{k)}$ of the algebraic extension multiplying them interpreted as real numbers $e_R^{k)}$

$$\begin{aligned} \prod_p N_p(|\hat{\zeta}|^2) &= \sum_k e_R^{k)} \prod_p N_p(\frac{1}{|\hat{\zeta}|_k^2}) = |\zeta|^2 \ , \\ |\hat{\zeta}|^2 &= \sum_k e^{k)} |\zeta|_k^2 \ . \end{aligned} \tag{6}$$

From this formula Universality Principle follows automatically. Since $|\hat{\zeta}|^2$ can be regarded as a vector having infinite number of components, the only manner to achieve the vanishing of $\prod_p N_p(|\hat{\zeta}|^2)$ is to require that it contains a vanishing rational factor. As will be found, the points at which infinite number of the factors of $|\hat{\zeta}|^2$ can be rational, very probably belong to the lines $Re(s) = n/2$. Thus the Universality Principle, and as it seems, also Riemann hypothesis, reduces to the statement that the modified Adelic formula defines a genuine norm which vanishes only when the vector is a null vector and is equal to $|\zeta|^2$. Of course, one could consider also the possibility that this norm is proportional to $|\zeta|^2$.

0.2.2 The conditions guaranteing the rationality of the factors $|Z_{p_1}|^2$

Universality Principle states that zeros of ζ correspond to zeros of $|\hat{\zeta}|^2$. This quantity, when well-defined, belongs to an infinite-dimensional real algebraic extension of rationals, and its vanishing is possible if it contains a vanishing rational factor which is product of an infinite number of factors Z_{p_1} which are rational. $|\hat{\zeta}|^2$ is the product of the factors

$$\frac{1}{Z_{p_1}(x+iy)Z_{p_1}(x-iy)} = 1 - 2p_1^{-x} Re[p_1^{iy}] + p_1^{-2x} \ . \tag{7}$$

This expression equals to a rational number q, if one has

$$Re[p_1^{iy}] = \frac{qp_1^x - p_1^{-x}}{2} \ . \tag{8}$$

In this case the integer multiples ny do not satisfy the rationality condition, to say nothing about the superpositions of different values of y. It is also implausible that this condition would hold true for an infinite number of primes p_1 required by the vanishing of a rational factor of $\hat{\zeta}$.

An alternative manner to achieve rationality is by requiring that the two terms are separately rational. p_1^{-2x} factor is rational only if one has $x = n/2$. To achieve rationality $Re[p_1^{iy}]$ should contain a factor compensating the irrationality of the $p_1^{-n/2}$ factor somehow. On the lines $Re[s] = x = n/2$ one has

$$\frac{1}{Z_{p_1}(n/2+iy)Z_{p_1}(n/2-iy)} = 1 - 2p_1^{-n/2}Re[p_1^{iy}] + p_1^{-n} \ .$$

It is of crucial importance that the moduli squared depend on the real part of p_1^{iy} only. If this is rational, rationality is achieved for even values of n.

On the lines $Re[s] = n + 1/2$ rationality is achieved provided that p_1^{iy} factors contain the phase factor $(r_1 + is_1\sqrt{k})/\sqrt{p_1}$ compensating the $p_1^{-1/2}$ factor and multiplying a factor which of the same type:

$$\begin{aligned} p_1^{iy} &= U_1 U = \frac{(r_1 + is_1\sqrt{k})}{\sqrt{p_1}} \times \frac{(r+is\sqrt{k})^2}{r^2+s^2k} \ , \\ r_1^2 + s_1^2 k_1 &= p_1 \ . \end{aligned} \tag{9}$$

The latter equation is satisfied if one has

$$k = \sqrt{p_1 - m^2} \ , \ 0 < m < \sqrt{p} \ . \tag{10}$$

On the lines $Re[s] = n$ one must have

$$p_1^{iy} = \frac{(r+is\sqrt{k})^2}{r^2+s^2k} \ . \tag{11}$$

The overall conclusions are following.

a) The vanishing of $|\hat{\zeta}|^2$ requires only the rationality of the *real parts* of Z_{p_1} for infinite number of values of p_1. The basic ansatz allows rationality only on the lines $Re[s] = n/2$ and my subjective feeling is that it is extremely implausible that exceptional ansatz gives rise to the rationality of an infinite number of $|Z_{p_1}|^2$ factors. That this is really the case might turn out to be difficult part in attempts to prove Riemann hypothesis even if one has proved the identity $\prod_p N_p(|\hat{\zeta}|^2) = |\zeta|^2$ and that this product defines a norm.

b) Rationality requirement allows p_1^{-iy} to consist of the products of the phases of very general algebraic numbers $r + is\sqrt{k}$. The products of these numbers are always of same form and their norm squared is $r^2 + s^2k$. Geometrically these numbers correspond to orthogonal triangles with one or two sides having integer valued length and remaining side having integer valued length squared.

c) For given value of y all integer multiples ny of y provide a solution of the rationality conditions. It is not necessary to require that the algebraic extensions $r + is\sqrt{k(p_1, y_i)}$ associated with y_1 and y_2 satisfying the condition, are same for given value of p_1: that is, one can have

$$k(p_1, y_1) \neq k(p_1, y_2) \ .$$

For $k(p_1, y_1) = k(p_1, y_2)$ also the linear combinations $m_1y_1 + n_1y_2$ satisfy rationality conditions. For the minimal solution to the rationality conditions, only multiples of each y solve the rationality conditions. For the maximal solution all solutions y_i correspond to the same algebraic extension for given p_1 and unrestricted linear superposition of the y_i holds true.

d) For $p \ mod \ 4 = 1$ rational phase factors p_1^{-iy} defined by the powers of the Gaussian primes provide the minimal manner to achieve rationality such that unrestricted superposition of solutions holds true. For $p_1 \ mod \ 4 = 3$ and $p_1 \ mod \ 3 = 1$ the minimal manner to achieve compensation is by using Eisenstein primes. For the primes $p_1 \ mod \ 4 = 3$ and $p_1 \ mod \ 3 = 2$ one cannot compensate $\sqrt{p_1}$ factor using Gaussian or Eisenstein primes and a more general algebraic extension of integers is necessary. For given prime p_1 there is finite number of possible algebraic extensions.

0.2.3 The conditions guaranteing the reduction of the p-adic norm

The term p_1^{-iy} appearing in the factors $1/Z_{p_1}$ is inversely proportional to integers and thus have p-adic norm which is larger than one for the primes appearing as factors of the integer n_1. Some mechanism guaranteing the reduction of the p-adic norm must be at work and this mechanism gives strong conditions on the allowed phases p_1^{iy}.

The condition guaranteing the reduction is very general. What is required is the reduction of the p-adic norm

$$|X\overline{X}|_p \ , \quad X = 1 - Up_1^{iy} \ , \quad U = (\epsilon p_1)^{-n/2} \ . \tag{12}$$

Here one has $\epsilon = 1$ for even values of n whereas for for odd values of n one has $\epsilon = \pm 1$ depending on whether the square root exists or not p-adically: the sole purpose of this factor is to take care that the p-adic counterpart of U is an ordinary p-adic number.

By writing

$$p_1^{-iy} \equiv cos(\phi) + isin(\phi) \ ,$$

one obtains

$$|X\overline{X}|_p = |1 - 2Ucos(\phi) + U^2|_p \ .$$

Not surprisingly, the vanishing of the norm modulo p implies in modulo p accuracy

$$U = cos(\phi) - \sqrt{-1}sin(\phi) \ .$$

Since U must be real, the only possible manner to satisfy the condition is to require that

$$sin(\phi) = 0 \ mod \ p \ , \quad cos(\phi) = 1 \ mod \ p \ . \tag{13}$$

Clearly, ϕ must correspond to angle 0 or π in modulo p accuracy. What this condition says is that partition functions Z_{p_1} are real in order p. This is very natural condition on the line $Re[s] = 1/2$ where the ζ is indeed real.

The condition $cos^2(\phi) = 1 \ mod \ p$ implies

$$p_1^n \ mod \ p = 1 \ . \tag{14}$$

p_1 can be always written as a power $p_1 = a^k$ of a primitive root a satisfying $a^{p-1} = 1$ modulo p such that k divides $p-1$. Thus $p_1^n \ mod \ p = 1$ holds true only only if $n \ mod \ (p-1)/k = 0$ is satisfied.

The conditions guaranteing modulo p reality of Z_{p_1} for prime p dividing the denominator of p_1^{-iy}, when written explicitely, give

$$\begin{array}{lll} Re[s]=n\ : & r^2-s^2k=r^2+s^2k\ , & \frac{2rs}{r^2+s^2k}=0\ , \\ Re[s]=n+\frac{1}{2}\ : & (r^2-s^2k)r_1-2rss_1k=r^2+s^2k\ , & \frac{2rsr_1+(r^2-s^2k)s_1}{r^2+s^2k}=0\ . \end{array} \quad (15)$$

In the case of Gaussian primes ($k=1$) also second option is possible since the multiplication with $\pm i$ yields new rational phase factor: this option corresponds simply the exchange of r^2-s^2 and $2rs$ factors in the formula above.

Rather general solution to the conditions can be written rather immediately. In both cases the conditions

$$s\ mod\ p^2=0\ ,\quad r\ mod\ p=0 \quad (16)$$

are satisfied. Note that $s\ mod\ p^2=0$ is necessary since $r^2+s^2k\ mod\ p=0$ holds true. Besides this the conditions

$$\begin{array}{ll} r_1^2+s_1^2k\ mod\ p=1 & \text{for}\ \ Re[s]=n\ , \\ s_1\ mod\ p=0\ \&\ r_1\ mod\ p=1 & \text{for}\ \ Re[s]=n+\frac{1}{2}\ , \end{array} \quad (17)$$

are satisfied.

If p_1^{-iy} is inversely proportional to integer containing as factors powers of a prime p larger than p_1, the reduction of the norm cannot occur for $Re[s]=1/2$ but is possible for sufficiently large values of $Re[s]=n/2$. For $p_1=2$ and $p_1=3$ factors the reduction of the norm is certainly not possible on the line $Re[s]=1/2$ since the condition $2p+1\leq p_1$ cannot be satisfied for any prime in these cases. The reduction of the p-adic norm of the ζ suggests strongly that the condition $2p_i+1\leq p_1$ is satisfied for large primes p_1. If it is satisfied completely generally, the phase factors associated with Z_2 must be of the general form

$$2^{-iy} = \frac{(\pm 1 \pm i)}{\sqrt{2}} \times \frac{(m(y) + i\sqrt{2^{2n} - m^2(y)})}{2^n} \ .$$

This constraint and similar constraints associated with larger primes give very strong constraints on the zeros.

The general conclusions are following.

a) The reduction of the p-adic norm and the related modulo p reality of Z_{p_1} is the p-adic counterpart for the reality of ζ on the critical line which suggests that it might occur completely generally. It requires that $p_1^n \, mod \, p = 1$ holds true for all primes appearing as factors of the denominator n_1 of the rational part of the phase p_1^{-iy}.

b) If the denominator of p_1^{-iy} is squarefree integer, the p-adic norm of Z_{p_1} is never larger than unity except possibly in the diagonal case $p = p_1$.

c) In the diagonal case the norm grows like p_1^{n+1} for $Re[s] = n + 1/2$ and p_1^n for $Re[s] = n$. This conforms with the fact that ζ has no zeros for $Re[s] \geq 1$ but has zeros for $Re[s] = -2n$.

d) If rational points of ζ obey linear superposition, then the rational points on the lines $Re[s] = n$ contain an even number of y_i:s needed to achieve the rationality of $Re[p^{-iy}]$. Hence the denominator tends to have larger p-adic norm than it can have on the line $Re[s] = 1/2$. This means that the line $Re[s] = 1/2$ is optimal as far as zeros of $|\hat{\zeta}|^2$ are considered. It can however happen that in the product $p_1^{iy_1} p_1^{iy_2}$ complex conjugates of factor phases can compensate each other so that the p-adic norm of $p_1^{i(y_1+y_2)}$ is not always larger than the norms of the factors. In particular, the factors $(r_1 + is_1\sqrt{k})/\sqrt{p_1}$ could cancel in the product $p_1^{iy_1} p_1^{-iy_2}$ This mechanism could explain the emergence of almost zeros $y_{ij} = y_i - y_j$ of ζ on the line $Re[s] = 1$ required by the inner product property of the Hermitian form defined by the superconformal model for the zeros of ζ.

0.2.4 Gaussian primes and Eisenstein primes

The general manner to satisfy the rationality requirement is to assume that the phases p_1^{iy} correspond to orthogonal triangles with one or two sides with an integer valued length and one side with integer valued length squared. A rather general and mathematically highly interesting manner to realize the rationality of the the phases $p_1^{-n/2} p_1^{iy}$ is by choosing the phases to be products of Gaussian or Eisenstein primes.

Gaussian primes consist of complex integers $e_i \in \{\pm 1, \pm i+\}$, ordinary primes $p\ mod\ 4 = 3$ multiplied by the units e_i to give four different primes, and complex Gaussian primes $r \pm is$ multiplied by the units e_i to give 8 primes with the same modulus squared equal to prime $p\ mod\ 4 = 1$. Every prime $p\ mod\ 4 = 1$ gives rise to 8 nondegenerate Gaussian primes. Pythagorean phases correspond to the phases of the squares of complex Gaussian integers $m + in$ expressible as products of even powers of Gaussian primes $G_p = r + is$:

$$G_p = r + is \ , \quad G\overline{G} = r^2 + s^2 = p \ , \ p \text{ prime } \& \ p \ mod \ 4 = 1 \ . \tag{18}$$

The general expression of a Pythagorean phase expressible as a product of even number of Gaussian primes is

$$U = \frac{r^2 - s^2 + i2rs}{r^2 + s^2} \ . \tag{19}$$

By multiplying this expression by a Gaussian prime i, one obtains second type of Pythagorean phase

$$U = \frac{2rs + i(r^2 - s^2)}{r^2 + s^2} \ . \tag{20}$$

Gaussian primes allow to achieve rationality of $p_1^{-n+1/2} p^{-iy}$ factor for $p_1\ mod\ 4 = 1$. The generality of the mechanism suggests that Gaussian primes should be very important. For $Re[s] \neq n/2$ it is not possible to achieve complex rationality with any decomposition of p_1^{iy} to Gaussian primes.

Besides Gaussian primes also so called Eisenstein primes are known to exist and the fact that only the rationality of the real parts of $1/Z_{p_1}$ factors is necessary for the rationality of $|Z_{p_1}|^2$ means that they are also possible. Note however that now the multiplication the phase by $\pm i$ makes the real part of the phase irrational, and is thus not allowed. Thus only four-fold degeneracy is present now for ζ.

Whereas Gaussian primes rely on modulo 4 arithmetics for primes, Eisenstein primes rely on modulo 3 arithmetics. Let $w = exp(i\phi)$, $\phi = \pm 2\pi/3$, denote a nontrivial third root of unity. The number 1-w and its associates obtained by multiplying this number by ± 1 and $\pm i$; the rational primes $p\ mod\ 3 = 2$ and its associates; and the factors $r + sw$ of primes $p\ mod\ 3 = 1$ together with their associates, are Eisenstein primes. One can write Eistenstein prime in the form

$$w = r - \frac{s}{2} + is\frac{\sqrt{3}}{2} \ . \tag{21}$$

What might be called Eisenstein triangles correspond to the products of powers of the squares of Eisenstein primes and have integer-valued long side. The sides of the orthogonal triangle associated with a square of Eisenstein prime E_p have lengths

$$(r^2 - rs - \frac{3s^2}{2} \ , \ s\frac{\sqrt{3}}{2} \ , \ p = r^2 + s^2 - rs) \ .$$

Eisenstein primes clearly span the ring of the complex numbers having the general form $z = (r + i\sqrt{3}s)/2$, r and s integers. To my very restricted best knowledge, the other algebraic extensions of integers do not allow the notion of prime number.

One can use Eisenstein prime E_p to achieve the replacement of the $p_1^{-1/2}$-factor with $1/p_1$-factor in the partition functions Z_{p_1} the same effect for $p_1\ mod\ 4 = 1$ and $p_1\ mod\ 3 = 1$ with the net result that $i\sqrt{3}$ term appears. This trick does not work for $p_1\ mod\ 4 = 3$ and $p_1\ mod\ 3 = 2$. Note that the presence of *both* Gaussian and Eisenstein primes in the same factor Z_{p_1} is not allowed since in this case also the real part of Z_{p_1} would contain $\sqrt{3}$. This suggests that quite generally $p\ mod\ 4 = 1$ *resp.* $p\ mod\ 4 = 3 \wedge p\ mod\ 3 = 1$ parts of $\hat{\zeta}$ could correspond to Gaussian *resp.* Eisenstein primes.

For the factors Z_{p_1} satisfying $p_1\ mod\ 4 = 3$ & $p_1\ mod\ 3 = 2$ simultaneously, neither Gaussian nor Eisenstein primes can affect the rationalization of $p^{-n+1/2-iy}$ factor, and in this case more general algebraic extension of complex numbers is necessary as already found.

0.3 Tests for the $|\hat{\zeta}|^2 = |\zeta|^2$ hypothesis

The fact that the phases p_1^{iy} correspond to nonvanishing values of y, suggests that $|\hat{\zeta}|^2 = |\zeta|^2$ equality holds on the real axis only in the sense of a limiting procedure $y \rightarrow 0$. If the the values of y giving rise to allowed phases obey linear superposition (that is $k_1(p_1, y)$ defining the algebraic extension does not depend on y), the allowed values of y form a dense set of the real axis, since arbitrarily small differences $y_i - y_j$ are possible for the zeros of ζ. Hence the limiting procedure $y \rightarrow 0$ should be well-defined and give the expected answer if the basic hypothesis is correct.

0.3.1 What happens on the real axis?

The simplest test for the basic hypothesis is to look what happens on the real axis at the points $s = n$. Real ζ diverges at $s = 1$ and $s = 0$ and has trivial zeros are at the points $s = -2n$. The norm of $\hat{\zeta}$ is given by

$$|\hat{\zeta}(n)|_R \;=\; \prod_p \left[\prod_{p_1} |1 - p_1^{-n}|_p \right] \; . \tag{22}$$

For $n = 0$ a straightforward substitution to the formula implies that $|\hat{\zeta}(0)|$ vanishes. For $n > 0$ one has

$$|\hat{\zeta}(n)|_R \;=\; \prod_p \left[\prod_{p_1} |\frac{p_1^n - 1}{p_1^n}|_p \right] = \prod_p p^n \left[\prod_k \prod_{p_1^n \; mod \; p^k = 1} p^{-k} \right] \; . \tag{23}$$

Since the number of primes p_1 satisfying the condition $p_1^n \; mod \; p^k = 1$ is infinite, the norm vanishes for all values $n > 0$. For $s = -n < 0$ one has,

$$|\hat{\zeta}(n)|_R \;=\; \prod_p \left[\prod_{p_1} |1 - p_1^{n}|_p \right] \; . \tag{24}$$

and also this product vanishes always.

How to understand these results?

a) The results are consistent with the view that $|\zeta|_R$ on the real axis should be estimated by taking the limit $y \rightarrow 0$. Since the values of y in question involve necessarily differences of very large values of y, it is conceivable that the limiting procedure does not yield zero. That the limiting procedure can give zero for $Re[s] < 0$ could be partially due to the fact that for $Re[s] = -n < 0$ one has for the diagonal $p_1 = p$ contribution $|Z_p(-n + iy)|_p = 1$

whereas for $Re[s]=n>0$ one has $|Z_p(n+iy)|_p>1$ in general. Furthermore, for $Re[s]=-n$ only $p_1^n\ mod\ p^k=1$ condition leads to the reduction of the p-adic norm of $Z_{p_1\neq p}$ whereas for $Re[s]=-2n$ also $p_1^n\ mod\ p^k=-1$ condition has the same effect.

b) One cannot exclude the possibility that only the proportionality $|\hat{\zeta}|^2 \propto |\zeta|^2$ holds true. For instance, in the superconformal model predicting that the physical states of the model correspond to the zeros of ζ on the critical line, the Hermitian form defining the 'inner product' is proportional to the product of $sin(i\pi z)\Gamma(z)\zeta(z)$. This function vanishes for $Re[s]\notin\{0,1\}$ and the coefficient function of ζ is finite in the critical strip. For $s=0$ this function however has the value $-1/2$ and for $s=1$ the value is 1, whereas the naively evaluated value of $|\hat{\zeta}|$ vanishes identically at these points. Thus something else is necessarily involved.

c) It could also be that the product represention for the norm squared of $\hat{\zeta}$ as a product of its p-adic norms converges only in a restricted region. It would not be surprising if the negative values of y were excluded from the region of convergence for the representation of $|\hat{\zeta}|^2$ as a product of its p-adic norms. Concerning the proof of the Riemann hypothesis, the minimal requirement is that the region $[1/2\leq Re[s]\leq 1\ ,\ y\neq 0]$ is included in the region of convergence.

One might think that $|\zeta|^2=|\hat{\zeta}|^2$ hypothesis is testable simply by comparing the norm squared of the real zeta with the product of the p-adic norms of $|\hat{\zeta}|^2$. The problems are that the value for the product of p-adic norms is extremely sensitive to numerical errors since the p-adic norm of Pythagorean triangles phases fluctuates wildly as a function of the phase angle, and that one does not actually know what the values of p_1^{iy} actually are. One testable prediction, also following from the superconformal model of the Riemann Zeta, is that the superpositions of the zeros are probably almost zeros or minima of $|\zeta|_R$ on the lines $Re[s]=n/2$. One could also try to understand whether the the norm of $\hat{\zeta}$ allows a continuation to a continuous function of the complex argument identifiable as a modulus of an analytic function.

0.3.2 Can the imaginary part of $\hat{\zeta}$ vanish on the critical line?

Riemann Zeta is real on the critical line $Re[s]=1/2$. A natural question is whether also $\hat{\zeta}$ has a vanishing imaginary part on this line. This is certainly not necessarily since $\hat{\zeta}$ has values in the infinite-dimensional algebraic extension of rationals. One cannot formulate the vanishing condition for the imaginary part in terms of the norm squared of any quantity defined by using the generalization of the adelic formula.

The properties of $\hat{\zeta}$ must be however consistent with the vanishing of $Im[\zeta]$ on the critical line. The reality of ζ on the critical line follows from the symmetry with respect to the critical line reducing on the critical line to the condition $\zeta(s) = \zeta(1-s)$ implying the reality of $\zeta(s)\zeta(1-s)$. This condition makes sense also for $\hat{\zeta}$. In general case, one has

$$\hat{\zeta}(s)\hat{\zeta}(1-s) = \prod_{p_1} Z_{p_1}(x+iy)Z_{p_1}(1-x-iy) = \prod_{p_1} \frac{1}{\left[1 - p_1^{-x}p_1^{-iy} - p_1^{-1+x}p_1^{iy} + \frac{1}{p_1}\right]} \ .$$

Due to the presence of p^{-x} terms, the moduli squared for these factors are complex irrational numbers.

On the line $Re[s] = 1/2$, the product representation for this function reduces to the product of *real* factors

$$\frac{1}{Z_{p_1}(1/2+iy)Z_{p_1}(1/2-iy)} = 1 - p_1^{-1/2}(p_1^{iy} + p_1^{-iy}) + \frac{1}{p_1} \tag{25}$$

in the algebraic extension of rationals. Thus the reality and rationality of the function $\hat{\zeta}(s)\hat{\zeta}(1-s)$ on the critical line corresponds in a very transparent manner the reality of ζ on the critical line. Note also that the modulo p reality of the factors Z_{p_1} implied by the reduction of the p-adic norm can be regarded as the p-adic counterpart for the reality of ζ on the critical line.

0.3.3 What about non-algebraic zeros of ζ?

In principle real ζ can also have irrational zeros z. The following argument however demonstrates that they do not pose a problem.

a) Pythagorean phase condition means that only the linear combinations

$$y = \sum n_k y_k \tag{26}$$

with n_k integer and p^{iy_k} complex rational phase for every p, are possible. Since the differences $y_i - y_j$ are known to have arbitrarily small values for the zeros of ζ, this means that the values allowed by the rationality requirement of $\hat{\zeta}$ of y must form a dense subset of reals.

b) If the norm of $\hat{\zeta}$ defined as a product of its p-adic norms indeed equals to the norm of the real ζ, one obtains strict bounds for the norm of the real ζ excluding the zeros in the dense set inside the critical strip. The continuity of the real ζ in turn implies that it cannot vanish except on the critical line.

0.4 Riemann Zeta and quantum TGD

The idea that quantum TGD could be regarded as a generalized number theory was stimulated by the notion of infinite primes inspired by TGD inspired theory of consciousness and by the notion ofe phase preserving canonical identification playing a key role in the understanding of the p-adic aspects of quantum TGD. The work with Riemann hypothesis has led to a realization that this vision might be realized in a unexpectedly concrete sense.

0.4.1 Phase preserving canonical identification and Riemann Hypothesis

The sharpened form of Riemann hypothesis has interesting potential implications concerning the construction of quantum TGD. For given p, the p-adic norms of Pythagorean phases p_1^{iy} fluctuate wildly as the real coordinate y varies in an arbitrarily small range. The proposed hypothesis however means that in the set of the allowed phases p_1^{iy} ζ there must be a long range order removing this wild variation and making p-adic ζ functions effectively continuous functions of y. A possible explanation is that the very fact that p_1^{iy} defines Pythagorean phase for infinite number of primes p_1, implies this effective continuity.

The existence of only two values y_i for which $p_1^{-iy_i}$ are rational for infinite number of primes p_1, implies that the set $y = n_1y_1 + n_2y_2$ defining Pythagorean phases forms a dense set on the real axis. Even for single value of y and single value of p_1 the phase factors p_1^{-iny} define a dense subset of all possible phase factors. This suggests that one could use Pythagorean phases to define the phase preserving canonical identification with Pythagorean phases to optimize the continuity for the mapping of the real physics to the p-adic physics. Thus the sharpened form of the Riemann hypothesis would become an essential part of quantum TGD and physics a la TGD.

0.4.2 Zeros of ζ code for infinite energy quantum states of TGD Universe

The first amazing analogy with TGD is the structure of the partition functions Z_{p_1}. These partition functions are proportional to integer $n_1 = \prod_i p_i^{n_i}$ such that $p_1 \, mod \, p_i = 1$ holds true. This requires $2p_i + 1 \leq p_1$. TGD inspired interpretation is that Z_{p_1} corresponds to a spacetime sheet with p_1-adic effective topology to which particles represented by smaller spacetime sheets with effective p_i-adic topologies are 'glued'. The square free integer $n = \prod_i p_i$ codes the presence of a fermion in mode p_i on spacetime sheet with effective topology characterized by p_1. The reduction of the p-adic norm from p_i to p_i^{-n} corresponds to the presence of $n+1$ bosons associated with spacetime sheet(s?) with p_i-adic topology. One can also identify the factors of partition function representing purely bosonic states. If p does not divide the integer associated with the phase p_1^{-iy} but the norm of state is reduced by power p^{-k}, the spacetime sheet p_1 contains k bosons of type p but nor fermions of type p.

The generation of Bose-Einstein condensates of bosons in the mode is thus necessary for the reduction of the p-adic norm. The exponent $k = N_B - N_F$ of p-adic norm p^{-k} of $\hat{\zeta}$ codes the net energy of bosons and fermions on the spacetime sheet with fermions giving negative contribution to the energy.

When $|\hat{\zeta}|^2$ vanishes, the state described by $\hat{\zeta}$ contains infinite number of bosons and has infinite energy. The zeros of $\hat{\zeta}$ correspond to quantum states of infinite energy (macroscopic quantum states) and the value of y actually codes the particle content of the quantum state. Thus it seems that ζ could provide a toy model for the quantum critical dynamics of quantum TGD. In fact, the following considerations suggest that much more than a mere toy model might be in question. Also the spectroscopy of quantum TGD might have number theoretic interpretation.

This coding applies also in case of arithmetic quantum field theory when one interprets it as a collection all p-adic p-adic quantum field theories and interprets Riemann Zeta and closely related L-functions as representations of the physical states. In fact one can divide these functions with Z_{p_1} and multiply them with powers of Z_{p_1} to get more general states with the same zeros but different particle content. p_1 codes for spacetime sheets containing particle like spacetime sheets and for a given value of p the factors Z_{p_1} of the p-adic ζ code for the numbers of fermions and bosons in mode p topologically condensed on the spacetime sheet labelled by p_1.

0.4.3 Zeros of ζ code also elementary particle numbers

The properties of Gaussian and Eisenstein primes have intriguing parallels with quantum TGD at the level of elementary particle quantum numbers.

a) The lengths of the complex vectors defined by the non-degenerate Gaussian and Eisenstein primes are square roots of primes as are also the preferred p-adic length scales L_p: this suggests a direct connection with quantum TGD.

b) Each nondegenerate (purely real or imaginary) Gaussian prime of given norm p corresponds to 8 different complex numbers $G = \pm r \pm is$ and $G = \pm s \pm ir$. This is the number of different spin states for the imbedding space spinors and also for the color states of massless gluons (note that in TGD quark color is not spinlike quantum number but is analogous to orbital angular momentum). Complex conjugation might be interpreted as a representation of charge conjugation and multiplication by $\pm 1, \pm i$ could give rise to different spin states. The 4-fold degeneracy associated with the $p \ mod\ 4 = 3$ Gaussian primes could correspond to the quartet of massless electroweak gauge bosons with a given helicity $[(\gamma, Z^0) \leftrightarrow \pm p)$ and $(W^+, W^-) \leftrightarrow \pm ip]$.

c) For Eisenstein prime E_{p_1} the multiplication by $\pm i$ does not respect the rationality of the real part of $|Z_{p_1}|^2$ and the number of states is reduced to four. Same restriction applies quite generally to the case when p_1^{iy} is of general form $r + i\sqrt{k}$. On the other hand, the effect of multiplication of Einstein primes E_{p_i} by these units remains invisible in the part of phase consisting of even powers of Einstein primes. Therefore the 8-fold degeneracy is effectively there and the interpretation is that this multiplication, although it changes the physical state, is not visible in the properties of the partition function.

d) The basic character of the quark color is triality realized as phases w which are third roots of unity. The fact that the phases are associated with the Eisenstein primes suggests that they might provide a representation of quark color. One can indeed multiply any Eisenstein prime in the product decomposition by factor 1, w or $\overline{w}$ and the interpretation is that the three primes represent three color states of quark. The obvious interpretation is that each factor Z_{p_1} with $p_1 \ mod\ 4 = 1$ could represent 8 possible leptonic states. Each factor Z_{p_1} satisfying $p_1\ mod\ 4 = 3$ and $p_1\ mod\ 3 = 1$ conditions simultaneously would correspond to a product of Eisenstein prime with Eisenstein phase and each prime p_i associated with Eisenstein phase would correspond to one color state of quark. Even a number theoretical counterpart of color confinement could be imagined.

There is also a further interesting analogy supporting the idea about number theoretical counterpart of the quark color. ζ decomposes into a product $\zeta_1 \times \zeta_3$, such that ζ_1 is the product of $p\ mod\ 4 = 1$ partition functions and ζ_3 the product of $p\ mod\ 4 = 3$ partition functions. This decomposition reminds of the leptonic color singlets and color triplet of quarks. Rather interestingly, leptons and quarks correspond to Ramond and Neveu-Schwartz type super Virasoro representations and the fields of N-S representation indeed contain square roots of complex variable existing p-adically for $p\ mod\ 4 = 3$.

e) What about the most general factors $r + is\sqrt{k}$? Can one assign some kind of color degeneracy also with these factors? It seems that this is the case. One can always find phase factors of type $U_\pm = (r \pm is\sqrt{k})/n$ with minimal values of n ($r^2 + s^2k = n^2$). The factors $1, U_\pm$ clearly give rise to a 3-fold degeneracy analogous to color degeneracy.

0.4.4 Gaussian and Eisenstein versions of infinite primes

The vision about quantum TGD as a generalized number theory generates also a third line of thoughts.

a) As has been found, the zeros of ζ code for the physical states of a supersymmetric arithmetic quantum field theory. As a matter fact, the arithmetic quantum field theory in question can be identified as arithmetic quantum field theory in which single particle states are labelled by Gaussian primes. The properties of the Gaussian primes imply that the single particle states of this theory have 8-fold degeneracy plus the four-fold degeneracy related to the $\pm i$ or ± 1-factor which could be interpreted as a phase factor associated with any $p\ mod\ 4 = 3$ type Gaussian prime. Also Eisenstein primes could allow the construction of a similar arithmetic quantum field theory.

b) The construction of the infinite primes reduces to a repeated second quantization of an arithmetic quantum field theory. A straightforward generalization of the procedure allows to define also the notion of infinite Gaussian and Eisenstein primes. Since each infinite prime is in a well-defined sense a composite of finite primes playing the role of elementary particles, this would mean that each composite prime in the expansion of an infinite prime has either four-fold degeneracy or eight-fold degeneracy. The interpretation of infinite primes could thus literally be as many-particle states of quantum TGD. In TGD the topology of spacetime surfaces of infinite size is characterized by infinite-p p-adic topology and the possibility of infinite-p p-, G- and E-adic topologies suggests the fascinating possibility that this infinite-p p-adic topology carries implicitly information about the discrete quantum numbers of all particles represented as spacetime sheets glued to the larger spacetime sheet.

0.4.5 G-adic and E-adic fractals?

A third line of thoughts relates to the possibility to generalize the notion of p-adicity so that could speak about G-adic and E-adic number fields. The properties of the Gaussian and Einsenstein primes indeed strongly suggest a generalization for the notion of p-adic numbers to include what might be called G-adic or E-adic numbers.

a) Consider for definiteness Gaussian primes. The basic point is that the decomposition into a product of prime factors is unique. For a given Gaussian prime one could consider the representation of the algebraic extension involved (complex integers in case of Gaussian primes) as a ring formed by the formal power series

$$G = \sum_n z_n G_p^n \quad . \tag{27}$$

Here z_n is Gaussian integer with norm smaller than $|G_p|$, which equals to p for $p \ mod \ 4 = 3$ and $\sqrt{p}$ for $p \ mod \ 4 = 1$.

b) If any Gaussian integer z has a unique expansion in powers of G_p such that coefficients have norm squared smaller than p, modulo G arithmetics makes sense and one can construct the inverse of G and number field results. For $p \ mod \ 4 = 1$ the extension of the p-adic numbers by introducing $\sqrt{-1}$ as a unit is not possible since $\sqrt{-1}$ exists as a p-adic number: the proposed structure might perhaps provide the counterpart of the p-adic complex numbers in case $p \ mod \ 4 = 1$. Thus the question is whether one could regard Gaussian p-adic numbers as a natural complexification of p-adics for $p \ mod \ 4 = 1$, perhaps some kind of square root of R_p, and if they indeed form a number field, do they reduce to some known algebraic extension of R_p?

c) In case of Eisenstein numbers one can identify the coefficients Z_n in the formal power series $E = \sum z_n E_p^n$ z_n as Eisenstein numbers having modulus square smaller than p associated with E_p and similar argument works aso in this case.

c) What is interesting from the physics point of view is that for $p\ mod\ 4 = 1$ the points G_p^n and E_p^n are on the logarithmic spiral $z_n = p^{n/2}exp(in\phi_0/2)$, where ϕ is the Pythagorean (Eisenstein) phase associated with G_p^2 (E_p^2). The logarithmic spiral can be written also as $\rho = exp(nlog(p)\phi/\phi_0)$. This reminds strongly of the logarithmic spirals, which are fractal structures frequently encountered in self-organizing systems: perhaps G- and E-adics might provide the mathematics for the modelling of these structures.

0.5 Discussion of other ideas related to the Riemann hypothesis

The study of the Riemann hypothesis has generated several competing candidates for the Great Principle. The progress made possible by the realization of the connection between Pythagorean phases and Gaussian primes allows to reduce the number of the competitors. It seems that the Universality Principle, together with the hypothesis that the norm of the analytically continued ζ equals to the product of the p-adic norms of $\hat{\zeta}$, is the only surviver.

0.5.1 Does p-adic ζ at the limit $p \rightarrow \infty$ correspond to real ζ?

The naive idea that reals are obtained as the limit R_p, $p \rightarrow \infty$, suggests real ζ corresponds to the $p \rightarrow \infty$ limit for p-adic ζ. This idea does not conform with the results obtained. The norms of p-adic ζ functions are in general non-vanishing for the zeros of $\hat{\zeta}$ and there are reasons to believe that at $p \rightarrow \infty$ limit the norms approach to unity.

This argument can be articulated also by using the p-adic number fields associated with infinite primes. Since the conditions for the reduction of the norm involve always modulo arithmetics and since modulo arithmetics reduces to ordinary arithmetics for the infinite primes, the p-adic norm of the p-adic ζ in this case is unity. Only the allowance of the infinite primes in the product representation of ζ might change the situation in this respect.

More precise formulation of this argument is an interesting exercise about infinite primes. The first question is about what one means with infinite-p p-adic numbers. It is not at all clear whether this notion even makes sense.

a) The most straightforward generalization introduces infinite p-adic numbers as formal powers series with integers coefficients. The problem is that there seems to be no manner to definet the inverse of an infinite-p p-adic number in this approach. For instance, the p-adic inverse of an integer, seems impossible to define.

b) One can however consider the possibility of introducing infinite-p p-adics as the formal power series

$$P^{k_0} \sum_{n \geq 0} x_n P^n \ ,$$

of infinite prime P with rationals, reals, or p-adic numbers, instead of integers, appearing as a coefficient field. In this case one could define the inverse of an infinite prime and in the lowest order it is just the rational, real or p-adic inverse. The use of rationals as coefficient field seems especially natural. p-Adic norm would be simply

$$|x|_P = P^{-k_0}$$

in the representation where infinite p-adic number is decomposed to a product of power of P and number with a unit norm. Norm is infinite or infinitesimal for $k_0 \neq 0$. One could also define the norm as

$$|x|_P = P^{-k_0}|x_0|$$

but this would mean the loss of the ultrametricity. One would however have ultrametricity for a sum of numbers with different P-adic norm.

This construction conforms with the construction of the generalized reals based on the idea of regarding infinite integers as spanning algebraic extension of reals (playing the role of imaginary units) with reals appearing as a coefficient field. One could also consider the possibility of using p-adic numbers as coefficient fields for infinite integers.

If one use defines infinite-p p-adics using rationals appearing as a coefficient field, it seems very difficult to imagine how to define roots $p_1^{m/n}$ without introducing an infinite-dimensional algebraic extension allowing all roots of all primes. Hence one must allow effectively rationals algebraically extended with all roots of all primes.

Does infinite-p p-adic ζ correspond to real ζ? The asnwer seems to be a definitely 'No' unless one defines the p-adic norm of infinite-p p-adic number in a non-ultrametric manner as $|x|_P = P^{-k_0}|x_0|$. Can infinite-p ζ have zeros if ultrametric norm is used? Unless one allows the presence of infinite primes p_1 in the product representation of ζ, the answer is negative. The reason is that the reduction of the p-adic norm for the factor Z_{p_1} occurs only if the condition $p_1^n\ mod\ p = 1$, n some positive integer, holds true. This condition cannot be satisfied if p is an infinite prime. Thus for infinite primes p-adic ζ has no zeros. Situation changes if one allows infinite primes p_1 in the representation of ζ. In real case their presence would imply that the product representation of ζ vanishes for $Re[s] < 0$. For $Re[s] > 0$ these factors give infinitesimal contribution to the value of the real ζ.

0.5.2 Does Local-Global Principle make sense?

Local-Global Principle has been the basic guideline for a long time in the attempts to understand the Riemann hypothesis. This hypothesis states that the zeros of the real ζ are also the zeros of the p-adic ζ functions (in the manner as they are defined). It must be emphasized that Universality Principle has same implications as this hypothesis.

How could one understand the hypothesis that each real zero of ζ corresponds to a p-adic zero? The simplest testable assumption guaranteing this is that the p-adic norm of ζ for any value of p is never larger than than the norm of the real zeta in case that p-adic zeta is defined:

$$\begin{aligned} &|\zeta(z = x + iy)|_p \leq |\zeta(z)|_R \text{ for all values of } p \ , \\ &x = \frac{m}{n} \text{ and } p_1^{iy} \text{ rational for every } p_1 \ . \end{aligned} \tag{28}$$

If this hypothesis holds true then each zero of the real ζ is zero of every p-adic ζ and the argument above works. In fact, a much weaker hypothesis stating that inequality holds for some values of p for given value of z is enough to lead to a plausibility argument in favour of the Riemann hypothesis.

From the foregoing it is clear that the Local-Global Principle is not consistent with the identification of the norm of the real ζ as a product of p-adic norms of ζ function. Also the fact that p-adic ζ functions need not vanish for the zeros of ζ violates the Local-Global Principle.