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NUMBER THEORY AS THE ULTIMATE PHYSICAL THEORY

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ABSTRACT

At the Planck scale doubt is cast on the usual notion of space-time and one cannot think about elementary particles. Thus, the fundamental entities of which we consider our Universe to be composed cannot be particles, fields or strings. In this paper the numbers are considered as the fundamental entities. We discuss the construction of the corresponding physical theory.

A hypothesis on the quantum fluctuations of the number field is advanced for discussion. If these fluctuations actually take place then instead of the usual quantum mechanics over the complex number field a new quantum mechanics over an arbitrary field must be developed. Moreover, it is tempting to speculate that a principle of invariance of the fundamental physical laws under a change of the number field does hold.

The fluctuations of the number field could appear on the Planck length, in particular in the gravitational collapse or near the cosmological singularity. These fluctuations can lead to the appearance of domains with non-Archimedean p-adic or finite geometry.

We present a short review of the p-adic mathematics necessary, in this context.

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The whole thing is a number
Pythagoras

1. - INTRODUCTION

Contemporary superstring theory is a beautiful theory which is capable of explaining many things about the elementary particles [for a review, see 1)]. String theory has accumulated many profound ideas such as the dual approach, the Kaluza-Klein approach, and supersymmetry, and to a considerable extent has realized the dreams of theorists about a unified theory and, in particular, it has avoided the appearance of divergences. In contrast to the usual field theory it does not deal with point-like objects, as does the local field theory, but with extended objects such as strings.

Nevertheless there are questions which are difficult to understand in string theory. In particular, gravitational collapse and the cosmological singularity are such problems. The present string theories do not tell us what to do with these problems. It seems that to understand these problems one needs to understand better what is space-time.

In string theory, as in other modern physical theories, the usual conception, going back to Euclid, Riemann and Einstein about space-time as a manifold with a metric, has been unquestioningly accepted. However, the superstring theory deals with very small distances of Planck order and it is very questionable that the standard structure of space-time, based on the notion of real numbers and the usual geometry, could indeed be applicable here. It seems also that a proper understanding of the problems of gravitational collapse and the cosmological singularity cannot be obtained in terms of the standard concept of space-time [for a discussion, see Ref. 2)].

Thus, one has to abandon the usual concept of space-time as well as the usual concept of quantum field theory or quantum string and must construct a new theory from which the usual quantum theory and the usual concept of space-time follow only in some macroscopic limit. Such a theory was proposed in a previous paper3) in which a hypothesis on the possible non-Archimedean structure of space-time on the sub-Planckian scale was considered and the corresponding dual approach was initiated. A possible role of the non-Archimedean p-adic space in physics was noted by Vladimirov and the author in Ref. 4).

In this paper this approach will be developed further. In particular it will be discussed in what sense numbers could be the fundamental entities of which we consider our Universe to be composed and the construction of the corresponding physical theory will be outlined. A hypothesis on the possible quantum fluctuations of the number field is also discussed. A short review of the necessary p-adic mathematics in this context is presented.

2. - QUANTUM GRAVITY, STRINGS AND NON-ARCHIMEDEAN GEOMETRY

Certainly, on distances of the Planck order, the effects of quantum gravity have to play a crucial rôle. It is known that gravitational measurement cannot be localized in a sub-Planckian domain [see the discussion of these questions in Refs. 5) and 6)]. More exactly it was shown in quantum gravity that in contrast to quantum electrodynamics[7)] principal limitations on field measurements arise which follow from the equivalence principle. In particular one has the inequality $\Delta\Gamma \gtrsim \ell_{P\ell}^2/\ell^3$ where $\Delta\Gamma$ denotes the uncertainty in determination of a connection coefficient, ℓ is the length scale, $\ell_{P\ell} = (\hbar G c^{-3})^{\frac{1}{2}}$ is the Planck length and G is Newton's constant.

Then there is an absolute limitation on length measurements $\ell \gtrsim \ell_{P\ell}$, so the Planck length is the smallest possible distance that can in principle be measured. The reason is the following. If we want to locate an elementary particle or string we need to have energy greater than the Planck mass $m_{P\ell} = (\hbar c G^{-1})^{\frac{1}{2}}$. But in such a case the corresponding gravitational field will have a horizon at $r = 2Gm_{P\ell}c^{-2} = 2\ell_{P\ell}$, shielding whatever happens inside the Schwarzschild radius. So no information of geometry in a sub-Planckian region is available. We see that in some sense a smallest quantum of space and time exists and it is of the order of the Planck scale. Really this means that we need a new theory instead of the usual quantum gravity or string theory in this high energy region.

Nevertheless, people sometimes hope that a satisfactory theory of quantum gravity can be found by means of a technical trick, i.e., one can hope to join the concept of Riemannian geometry with the quantum theory at a formal level. From our point of view the main problem with quantum gravity is not technical, but is a more fundamental one - we have to understand what quantum space-time on the Planck scale itself means. Note that a string is a curve in the usual space and therefore the string theory cannot help us here.

Indeed, what do we mean by usual space? In fact this mathematical notion expresses our point of view about the motion of solid objects. This question was cleaned up by Poincaré[8)] and Weyl[9)]. To good precision we can consider points, straight lines, planes, etc., to satisfy the axioms of Euclidean geometry (at least in small macroscopic distances). Let us stress that to actually realize these geometrical figures we have to deal with macroscopic bodies. So we can ask ourselves whether it is possible to construct a segment of a straight line, a plane, etc., from electrons and other elementary particles taken in small numbers in such a way that one can move them according to the axioms of Euclidean geometry. The answer is of course no. Therefore the usual meaning of space is inapplicable to them. The notion of space occurs only in the case when we deal with a large number of particles so this is a macroscopic notion.

Our intuitive understanding of properties of space is expressed in the axioms of elementary geometry. Let us consider elementary geometry. Because the geometrical postulates are intuitively self-evident it proved very difficult to compile a complete list of geometrical axioms. This goal was reached at the turn