A survey of the anabelian geometry of hyperbolic curves over finite fields

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k **finite** field

X/k smooth **hyperbolic** geo. connected curve type (g, r)

$k(X)$ function field

η base point over *generic* point $\quad \rightsquigarrow \quad \overline{k}/k \quad k(X)^{\mathrm{sep}}/k(X)$

$\overline{X} \stackrel{\mathrm{def}}{=} X \times_k \overline{k} \qquad \overline{k}(\overline{X})$

$G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k) =< \mathrm{Frob}_k > \qquad G_{k(X)} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(k(X)^{\mathrm{sep}}/k(X))$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_{\overline{k}(\overline{X})} & \longrightarrow & G_{k(X)} & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \pi_1(\overline{X}) & \longrightarrow & \pi_1(X) & \longrightarrow & G_k & \longrightarrow & 1 \end{array}$$

$$\rho_X : G_k \to \mathrm{Out}(\pi_1(\overline{X}))$$

Facts

(Pop/Harbater) $G_{\overline{k}(\overline{X})}$ is profinite **free**

Structure of $G_{k(X)}$ **unknown**

Structure of $\pi_1(\overline{X})$ and $\pi_1(X)$ **unknown**

X, Y smooth **hyperbolic** curves over **finite** fields k_X, k_Y

Theorem (Uchida 77)
The map

$$\mathrm{Isom}(k_X(X), k_Y(Y)) \to \mathrm{Isom}(G_{k_X(X)}, G_{k_Y(Y)})/\,\mathrm{Inn}\, G_{k_Y(Y)}$$

is a **bijection**

Theorem (Tamagawa 95 (**affine**) Mochizuki 04 (**proper**))
The map

$$\mathrm{Isom}(X, Y) \to \mathrm{Isom}(\pi_1(X), \pi_1(Y))/\,\mathrm{Inn}\,\pi_1(Y)$$

is a **bijection**

Consequence

$$\pi_1 \ : \{\text{hyp. curves/finite field}\} \hookrightarrow \{\pi_1(X)\}$$

Fact

$$\{\pi_1(X)\} \Leftrightarrow \{\pi_1(\overline{X}) \ + \ \rho_X : G_k \to \mathrm{Out}(\pi_1(\overline{X}))\}$$

Image of π_1-*map mysterious* ! !

$\mathfrak{P}\text{rimes} = \{\text{Prime integers}\} \qquad p = \text{char}(k)$

$\Sigma \subseteq \mathfrak{P}\text{rimes}$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\overline{X}) & \longrightarrow & \pi_1(X) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \pi_1(\overline{X})^{\Sigma} & \longrightarrow & \pi_1(X)^{(\Sigma)} & \longrightarrow & G_k & \longrightarrow & 1 \end{array}$$

$\Sigma = \mathfrak{P}\text{rimes} \setminus \{p\}$

$$1 \to \pi_1(\overline{X})^{(p')} \to \Pi_X \to G_k \to 1$$

$\Sigma = \{l \neq p\}$

$$1 \to \pi_1(\overline{X})^{l} \to \Pi_X^{(l)} \to G_k \to 1$$

$\Gamma_{g,r} \overset{\text{def}}{=}$ **profinite completion** of π_1^{top} Riemann surface of type (g, r)

Theorem (Grothendieck 60)

$$\pi_1(\overline{X})^{(p')} \simeq \Gamma_{g,r}^{(p')} \qquad \pi_1(\overline{X})^{l} \simeq \Gamma_{g,r}^{l} \qquad l \neq p$$

Fundamental Question

Reconstruct X from Π_X or $\Pi_X^{(l)}$?

X, Y smooth **hyperbolic** curves over **finite** fields k_X, k_Y

Theorem (Saïdi and Tamagawa 06)
The map

$$\mathrm{Isom}(X, Y) \to \mathrm{Isom}(\Pi_X, \Pi_Y)/\operatorname{Inn}\Pi_Y$$

is a **bijection**

Expect/Work in progress
If $\Sigma = \mathfrak{P}\mathrm{rimes} \setminus \{$ Finite subset$\}$. Then the map

$$\mathrm{Isom}(X, Y) \to \mathrm{Isom}(\Pi_X^{(\Sigma)}, \Pi_Y^{(\Sigma)})/\operatorname{Inn}\Pi_Y^{(\Sigma)}$$

is a **bijection**

Fundamental Question
Determine $\rho : G_k =< \mathrm{Frob}_k >\to \mathrm{Out}(\Gamma_{(g,r)}^{(p')})$ which arise from hyp. curve type (g, r) over the finite field k

Do not know if a pro-l version holds !!

X, Y smooth **hyperbolic** curves over **finite** fields k_X, k_Y

Birational Conjecture/Hom-form

The map

$$\mathrm{Mor}^{\mathrm{dom}}(X, Y) \to \mathrm{Hom}^{\mathrm{open}}(G_{k_X(X)}, G_{k_Y(Y)})/\operatorname{Inn} G_{k_Y(Y)}$$

is a **bijection**

Consequence

The map

$$\mathrm{Isom}(G_{k_X(X)}, G_{k_Y(Y)}) \to \mathrm{Surj}(G_{k_X(X)}, G_{k_Y(Y)})$$

is a **bijection**

Anabelian Conjecture/Hom-form

The map

$$\mathrm{Mor}^{\mathrm{dom}}(X, Y) \to \mathrm{Hom}^{\mathrm{open}}(\pi_1(X), \pi_1(Y))/\operatorname{Inn} \pi_1(Y)$$

is a **bijection**

Prime-to-characteristic Hom-form?

Strategy of Proof (Uchida / Tamagawa / Mochizuki)
Assume X, Y, **proper**

$$\sigma : \pi_1(X) \xrightarrow{\sim} \pi_1(Y)$$

1. Local theory
σ preserves the decomposition groups

$$\sigma(D_x) = D_y$$

Establish a **correspondence**

$$\phi : X^{\mathrm{cl}} \simeq Y^{\mathrm{cl}}$$

$$\sigma(D_x) = D_{\phi(x)}$$

2. Functions
Recover a **bijection**

$$\tau : k_X(X)^\times \xrightarrow{\sim} k_Y(Y)^\times$$

Kummer theory + Mochizuki's cuspidalization theory

3. Recover the additive structure
Show that τ is **additive**

$l \neq p$ **prime** integer

$U_S \stackrel{\text{def}}{=} X \setminus S$ **affine**

$\pi_1(U_S)^{\mathrm{c},l}$ = maximal **cuspidally** pro-l quotient of $\pi_1(U_S)$

$\stackrel{\text{def}}{=} \max\{\pi_1(U_S) \twoheadrightarrow H \twoheadrightarrow \pi_1(X) \ / \ \mathrm{Ker}(H \twoheadrightarrow \pi_1(X)) \text{ pro} - l\}$

Theorem (Mochizuki 04 /Hoshi 06)
If $S \stackrel{\text{def}}{=} \{x_1, ..., x_n\} \subset X(k_X)$ can reconstruct $\pi_1(U_S)^{\mathrm{c},l}$ from $\pi_1(X)$

Consequence
$S \subset X(k_X) \quad \leftrightarrow \quad T \stackrel{\text{def}}{=} \phi(S) = \{y_1, ..., y_n\} \subset Y(k_Y)$

$V_T \stackrel{\text{def}}{=} Y \setminus T$

$$\begin{array}{ccc} \pi_1(U_S)^{\mathrm{c},l} & \xrightarrow{\sigma^{\mathrm{c},l}_{(S,T)}} & \pi_1(V_T)^{\mathrm{c},l} \\ \downarrow & & \downarrow \\ \pi_1(X) & \xrightarrow{\sigma} & \pi_1(Y) \end{array}$$

$$\sigma : G_{k_X(X)} \twoheadrightarrow G_{k_Y(Y)}$$

$p = \mathrm{char}(k_X) = \mathrm{char}(k_Y)$

Definition (proper)

The homomorphism σ is called **proper** if $\exists$

$$\phi : X^{\mathrm{cl}} \to Y^{\mathrm{cl}}$$

with **finite fibres** such that:

$$\sigma(D_x) \underset{\text{open}}{\underset{=}{\subseteq}} D_{\phi(x)}$$

In particular:

$$\tau_{x,\phi(x)} : I_x^t \xrightarrow{\sim} I_{\phi(x)}^t \qquad (\xrightarrow{\sim} \hat{\mathbb{Z}}(1)^{(p')}\ )$$

$$M_X \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\hat{\mathbb{Z}}^{(p')}}(H^2(\pi_1(\overline{X}), \hat{\mathbb{Z}}^{(p')}), \hat{\mathbb{Z}}^{(p')}) \quad (\xrightarrow{\sim} \hat{\mathbb{Z}}(1)^{(p')}\ )$$

Definition (inertia-rigid)

σ is called **inertia-rigid** if $\exists\ \ \tau : M_X \xrightarrow{\sim} M_Y$ such that

$$\begin{array}{ccc} M_X & \longrightarrow & I_x^t \\ {\scriptstyle\tau}\big\downarrow & & \big\downarrow{\scriptstyle\tau_{x,\phi(x)}} \\ M_Y & \longrightarrow & I_{\phi(x)}^t \end{array}$$

commutes $\forall x \in X$

Theorem (Saïdi and Tamagawa 06)
There exists a natural map

$$\mathrm{Hom}(G_{k_X(X)}, G_{k_Y(Y)})^{\mathrm{sur,pr,inrig}} \to \mathrm{Hom}(k_Y(Y), k_X(X))$$

which induces a natural **bijection**

$$\mathrm{Isom}(G_{k_X(X)}, G_{k_Y(Y)}) \xrightarrow{\sim} \mathrm{Hom}(G_{k_X(X)}, G_{k_Y(Y)})^{\mathrm{sur,pr,inrig}}$$

Prime-to-p version ?

$$\Gamma_{k_X(X)} \stackrel{\mathrm{def}}{=} \text{geometrically prime-to-}p \text{ quotient of } G_{k_X(X)}$$

Definition

$$\sigma : \Gamma_{k_X(X)} \twoheadrightarrow \Gamma_{k_X(Y)}$$

is called **rigid** if

$$\sigma(D_x) = D_y, \quad \forall x \in X^{\mathrm{cl}}$$

Theorem (Saïdi and Tamagawa 06)
The natural map

$$\mathrm{Isom}(\Gamma_{k_X(X)}, \Gamma_{k_X(Y)}) \to \mathrm{Hom}^{\mathrm{sur,rig}}(\Gamma_{k_X(X)}, \Gamma_{k_X(Y)})$$

is a **bijection**.