COM3312: Logic and Computation

Specimen answers to the 2000 examination

1 Turing Machines

(a) The rule has the effect that once the machine has arrived in the halt state it stays there with no further changes.
It is needed so that T, U, and D are total functions, enabling them to be represented using function symbols in the logical representation.

(b) q is the current state.
a is the symbol on the currently scanned square.
b is the string of symbols running left from the scanned square as far as the leftmost non-blank square (or the empty string if there are no non-blank squares to the left of the scanned square.
c is the same as b but with 'left' replaced by 'right'.

(c)
$$\begin{array}{rcl}
q(n+1) & = & T(q(n),a(n)) \qquad \text{(The next state)} \\
a(n+1) & = & \begin{cases} head(b(n)) & \text{(if } D(q(n),a(n)) = -1 \text{ — moving left)} \\ U(q(n),a(n)) & \text{(if } D(q(n),a(n)) = 0 \text{ — staying still)} \\ head(c(n)) & \text{(if } D(q(n),a(n)) = 1 \text{ — moving right)} \end{cases} \\
b(n+1) & = & \begin{cases} tail(b(n)) & \text{(if } D(q(n),a(n)) = -1\text{)} \\ b(n) & \text{(if } D(q(n),a(n)) = 0\text{)} \\ U(q(n),a(n))^\frown b(n) & \text{(if } D(q(n),a(n)) = 1\text{)} \end{cases} \\
c(n+1) & = & \begin{cases} U(q(n),a(n))^\frown c(n) & \text{(if } D(q(n),a(n)) = -1\text{)} \\ c(n) & \text{(if } D(q(n),a(n)) = 0\text{)} \\ tail(c(n)) & \text{(if } D(q(n),a(n)) = 1\text{)} \end{cases}
\end{array}$$

(d) (i) $q(n) = 0$. (ii) $\exists n(q(n) = 0)$.

(e) The rules given above must be supplemented with general rules for handling numbers (for the states), lists (for b and c), etc. [See details in notes]. The premisses of the inference are all the above rules and a statement of the initial configuration, in the form

$$C(0) = (q(0), a(0), b(0), c(0)).$$

The conclusion is the statement that the machine eventually halts:

$$\exists a, b, c, i(C(i) = (0, a, b, c)).$$

If every case of the Entscheidungsproblem were solvable then it could be determined by mechanical means whether or not this inference was valid, i.e., whether or not the machine halts. But Turing had already shown that the Halting Problem is not solvable by mechanical means, and this therefore shows that the same is true of the Entscheidungsproblem.

2 Models and Interpretations

(a) An interpretation must specify a domain Δ (which may be any set), and it must map each element of the non-logical vocabulary of the language onto an element related to Δ, as follows:

an individual constant is mapped onto an element of Δ;
an n-ary function symbol is mapped onto a function from Δ^n to Δ;
an n-ary predicate symbol is mapped onto a subset of Δ^n.

There are general rules for determining the truth value of each sentence in the language relative to a given interpretation.
A model for a set of formula is any interpretation under which each formula in the set is true.

(b) Under $\mathcal{I}_1$,
A says that no number is less than itself;
B says that the 'less than' relation is transitive;
C says that of any two distinct real numbers, one is less than the other.
All these are obviously true.

(c) One possibility for $\mathcal{I}_2$ is as follows:

The domain is the power set of some set U.
P is interpreted as 'is a proper subset of'.

Then A is satisfied since no set is a proper subset of itself;
B is satisfied since the proper subset relation is transitive;
C is not satisfied, since, e.g., neither of $\{1, 2\}$ and $\{1, 3\}$ is a subset of the other.

For C to be a logical consequence of $\{A, B\}$ it is necessary that every model for $\{A, B\}$ satisfies C. So the existence of model $\mathcal{I}_2$, for $\{A, B\}$ which does not satisfy C proves that C is not a logical consequence of $\{A, B\}$.

- We need to find ain interpretation which satisfies B and C but not A. One possibility is:

  The domain is the set of all real numbers (as in $\mathcal{I}_1$);
  P is interpreted to mean 'is less than or equal to'.

  Then B and C are still satisfied, but A is not, since any number is less than or equal to itself. Hence A is not a logical consequence of $\{B, C\}$.

- Assume A and B.
  Suppose $P(x, y) \wedge P(y, x)$. Then by B, we have $P(x, x)$, contradicting A. Hence we cannot have both $P(x, y)$ and $P(y, x)$.
  Now suppose $P(x, y) \wedge x = y$. Then again we have $P(x, x)$, contradicting A, so we cannot have both $P(x, y)$ and $x = y$.
  Similarly, we cannat have both $P(y, x)$ and $x = y$.
  Hence we can have at most one of $P(x, y)$, $P(y, x)$, and $x = y$.

3 Proof Procedures

(a) A non-branching rule says that a formula ϕ of some specified form may be replaced by formulae ϕ_1 and ϕ_2 in the same branch. A rule of this kind means that ϕ logically implies the conjunction $\phi_1 \wedge \phi_2$. It is saying any model for a set of formulae including ϕ must also be a model for both ϕ_1 and ϕ_2.
A branching rule says that a formula ψ of some specified form may be replaced by formulae ψ_1 and ψ_2 on separate branches. A rule of this kind means that ψ logically implies the disjunction $\phi_1 \vee \psi_2$. It is saying that any model for a set of formulae including ψ must either be a model for ψ_1, or a model for ψ_2.

(b)

$$\begin{array}{c}
\begin{array}{l}
1.\ (\forall x P(x)) \rightarrow Q \\
2.\ \neg\forall x(P(x) \rightarrow Q) \\
\rule{3cm}{0.4pt}\ 2[x/a] \\
3.\ \neg(P(a) \rightarrow Q) \\
\rule{3cm}{0.4pt}\ 3 \\
4.\ P(a) \\
5.\ \neg Q
\end{array} \\
\rule{9cm}{0.4pt}\ 1 \\
\begin{array}{ll}
\begin{array}[t]{l} 6.\ \neg\forall x P(x) \\ \rule{2cm}{0.4pt}\ 6[x/b] \\ 7.\ \neg P(b) \end{array} & \qquad\qquad
\begin{array}[t]{l} 7.\ Q \\ \rule{1cm}{0.4pt}\ 5 \end{array}
\end{array}
\end{array}$$

There is nothing further that can be done. The left-hand branch is still open, leading to the model $\{P(a), \neg P(b), \neg Q\}$, which satisfies both 1 and 2. But 2 is the negation of the conclusion of the inference, which is therefore shown to be invalid.

(c)

$$\begin{array}{c}
\begin{array}{l}
1.\ (\forall x P(x)) \rightarrow Q(a) \\
2.\ \neg\exists x(P(x) \rightarrow Q(a))
\end{array} \\
\rule{9cm}{0.4pt}\ 1 \\
\begin{array}{ll}
\begin{array}[t]{l}
3.\ \neg\forall x P(x) \\
\rule{2cm}{0.4pt}\ 3[x/b] \\
5.\ \neg P(b) \\
\rule{2cm}{0.4pt}\ 2[x/b] \\
7.\ \neg(P(b) \rightarrow Q(a)) \\
\rule{2cm}{0.4pt}\ 7 \\
10.\ P(b) \\
11.\ \neg Q(a) \\
\rule{1cm}{0.4pt}\ 5, 10
\end{array} & \qquad\qquad
\begin{array}[t]{l}
4.\ Q(a) \\
\rule{2cm}{0.4pt}\ 2[x/a] \\
6.\ \neg(P(a) \rightarrow Q(a)) \\
\rule{2cm}{0.4pt}\ 6 \\
8.\ P(a) \\
9.\ \neg Q(a) \\
\rule{1cm}{0.4pt}\ 4
\end{array}
\end{array}
\end{array}$$

Both branches lead to a contradiction, so $\{1, 2\}$ is unsatisfiable, so the inference is valid.

(d) If the conclusion is false, then for every team member it is false that if they play well the team will win. This means that for every team member it is true that they play well but the team does not win. Hence everyone plays well but the team does not win, so the premiss is false.
Therefore, if the premiss is true, the conclusion is true too, so the inference is valid.
The mismatch between logic and intuition comes from the interpretation of ‘if’, which in this context suggests a notion of causality absent from the material conditional of the logic.
[In retrospect, I do not think this is a very satisfactory question: the issue is too ‘muddy’ to be properly handled within the scope of an examination answer.]

4 First-order Theories

(a) In Ex1, i must be 0, since for real numbers, $x + y = x$ if and only if $y = 0$.
In Ex2, i must be 1, since for real numbers, $x.y = x$ if and only if $y = 1$.
In Ex3, i must be $\emptyset$, since $\emptyset$ is the only set y for which we *always* have $x \cup y = x$.

(b) In (Ex1), (4) is not satisfied since, e.g., $2 + 2 \neq 2$.
In (Ex2), (4) is not satisfied since, e.g., $2 \times 2 \neq 2$.
In (Ex3), (4) is satisfied, since for any set X we have $X \cup X = X$.

(c)
$$\begin{aligned} f(x, f(x, y)) &= f(f(x, x), y) && \text{(by 2)} \\ &= f(x, y) && \text{(by 4)} \\ &= f(y, x) && \text{(by 1)} \\ &= f(f(y, y), x) && \text{(by 4)} \\ &= f(y, f(y, x)) && \text{(by 2)} \end{aligned}$$

(d) In (Ex1), (5) is satisfied since for any x we have $x + (-x) = 0$.
In (Ex2), (5) is not satisfied, since there is no y such that $0 \times y = 1$.
In (Ex3), (5) is not satisfied, since if $X \neq \emptyset$ then there is no set Y sich that $X \cup Y = \emptyset$.

(e)
$$\begin{aligned} f(x, f(x, y)) &= f(f(x, x), y) && \text{(by 2)} \\ &= f(x, y) && \text{(by 4)} \\ &= i && \text{(by assumption)} \\ f(x, f(x, y) &= f(x, i) && \text{(by assumption)} \\ &= x && \text{(by 3)} \end{aligned}$$

By (5), for each x there is a y such that $f(x, y) = i$, so by the above argument, $x = f(x, f(x, y)) = i$, so all elements are equal to i, i.e., the domain contains only one element.

(f) A complete axiomatisation of a first-order theory is a set of formulae whose logical consequences are precisely the formulae of the theory.
If $\{1, 2, 3\}$ were a complete axiomatisation of some theory, it would imply whichever of 4 and $\neg 4$ is in the theory. But (Ex1) satisfies $\{1, 2, 3, \neg 4\}$, while (Ex3) satisfies $\{1, 2, 3, 4\}$, so $\{1, 2, 3\}$ implies neither 4 nor $\neg 4$. [We could use 5 here instead of 4.]
The set $\{1, 2, 3, 4, 5\}$ only holds for a domain $\{i\}$ in which $f(i, i) = i$. All such domains are isomorphic, so we have a complete axiomatisation.

5 Essay Question

(a) The given description certainly applies to much (if not most) of what goes on in a computer, although it does not apply very comfortably to the ongoing computation of a reactive system such as an operating system. On the other hand, the process by which analogue informaion is digitised (e.g., in sound recording or scanning of images) might be described as a kind of computation, and yet it certainly does not come under the description given. I'm looking for a discussion of what processes should count as computational, of what processes can be described in the way suggested in the question, and of the relationship between these. [This material was discussed in the module as a preliminary to talking about the Turing machine model of computation.]

(b) I'm expecting some sort of account of what Gödel's theorems actually say, of the use that has been made of them people such as Lucas and Penrose to argue against the possibility of strong AI, and a critique or endorsement of those arguments.