COM3312

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING AND COMPUTER SCIENCE
Department of Computer Science

BSc Computer Science, Year 3 (option)

Logic and Computation

TWO HOURS

Answer THREE out of the five questions.

Candidates are advised to spend FORTY MINUTES on each question.

No electronic calculators of any sort are to be used during the course of this examination.

1. *Turing Machines* A Turing Machine has tape alphabet Σ and state set $Q = \{0, 1, 2, \ldots, n\}$, where 0 is the halt state and 1 the start state. The operation of the machine is specified by means of three functions

$$\begin{aligned} T: &\quad Q \times \Sigma \to Q \\ U: &\quad Q \times \Sigma \to \Sigma \\ D: &\quad Q \times \Sigma \to \{-1, 0, 1\} \end{aligned}$$

defining the next state, new symbol, and displacement for each state/symbol pair.

(a) The specification above is supplemented by the rule

$$\forall x \in \Sigma (T(0, x) = 0 \wedge U(0, x) = x \wedge D(0, x) = 0).$$

Explain the effect of this rule and why it is needed.

(2 marks)

(b) A configuration of the Turing machine at any computation step can be specified in a natural way as a quadruple $(q, a, b, c) \in Q \times \Sigma \times \Sigma^* \times \Sigma^*$. Explain the meaning of each element of such a quadruple, relating your explanation to the usual informal vocabulary of "tape", "scanned square", etc.

(4 marks)

(c) For a given initial configuration, the configuration after n steps is given by a quadruple $C(n) = (q(n), a(n), b(n), c(n))$ of the type specified in (b). Describe in detail how $C(n + 1)$ is determined from $C(n)$.

(7 marks)

(d) In terms of the function C introduced in (c), write down the conditions that

(i) the machine reaches the halt state after n computation steps;

(ii) the machine eventually reaches the halt state.

(2 marks)

(e) Explain in outline how the above considerations can be used to construct a first-order inference which is valid if and only if the machine halts. Explain the connection between the halting problem for Turing machines and the *Entscheidungsproblem* for first-order logic.

(5 marks)

2. *Models and Interpretations*

(a) Explain what is meant by an *interpretation* of a first-order language. What is the condition for an interpretation to be a *model* for a given set of formulae?

(2 marks)

The non-logical vocabulary of a language $\mathcal{L}$ consists of a single two-place predicate P. The logical vocabulary includes the identity symbol "=". In the remainder of the question we shall be interested in the following formulae of $\mathcal{L}$:

$$\begin{array}{ll} A: & \forall x \neg P(x,x) \\ B: & \forall x \forall y \forall z (P(x,y) \wedge P(y,z) \rightarrow P(x,z)) \\ C: & \forall x \forall y (P(x,y) \vee P(y,x) \vee x = y) \end{array}$$

(b) An interpretation $\mathcal{I}_1$ of $\mathcal{L}$ is defined as follows: the domain is the set of all real numbers, and the predicate P denotes the relation "is less than". For this interpretation, express in English the meaning of each of the formulae A, B, and C, and hence verify that all three formulae are satisfied by the interpretation.

(3 marks)

(c) Devise an alternative interpretation $\mathcal{I}_2$ for $\mathcal{L}$ which satisfies A and B but not C. Explain why this implies that C is not a logical consequence of the set $\{A, B\}$.

(5 marks)

(d) Use the same method to show that A is not a logical conseqence of $\{B, C\}$.

(4 marks)

(e) The formula C states that for each pair of domain elements at least one of three disjuncts must hold. Show that if A and B are satisfied, then *at most* one of the three disjuncts can hold for any pair of domain elements.

(6 marks)

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3. *Proof procedures* The *truth tree* proof procedure makes use of a fixed set of rewriting rules by which formulae are replaced or supplemented by new formulae constructed from them in accordance with the rules. There are two kinds of rule: branching rules and non-branching rules. The complete set of rules for first-order logic is listed in the appendix to this question paper.

(a) Explain the difference between the branching and non-branching rules, with reference to both what they mean and how they are applied.

(4 marks)

(b) Use the truth-tree method to show that the following inference is invalid, explaining exactly how you have shown this:

$$\frac{(\forall x P(x)) \to Q}{\forall x (P(x) \to Q)}$$

(6 marks)

(c) Use the truth-tree method to show that the following inference is valid:

$$\frac{(\forall x P(x)) \to Q(a)}{\exists x (P(x) \to Q(a))}$$

(6 marks)

(d) A possible interpretation of the inference in (c) is as follows:

If everyone plays well then the team will win.
Therefore there is someone such that, if (s)he plays well then the team will win.

Explain why, although this inference seems intuitively invalid, the result of section (c) holds for this interpretation in particular. What feature of the English version of this inference is mainly responsible for the mismatch between logic and intuition here?

(4 marks)

4. *First-order theories* The first-order theory of a commutative, associative operation f with identity i includes the following axioms:

$$\begin{array}{ll} (1) & \forall x \forall y (f(x,y) = f(y,x)) \\ (2) & \forall x \forall y \forall z (f(x, f(y,z)) = f(f(x,y),z)) \\ (3) & \forall x (f(x,i) = x) \end{array}$$

Three examples of such operations are

(Ex1)	Addition on the set of real numbers.
(Ex2)	Multiplication on the set of real numbers.
(Ex3)	Union on the set of subsets of the real numbers.

(a) For each of the examples (Ex1)–(Ex3), state, with justification, what element is denoted by the constant symbol i appearing in the axioms.

(3 marks)

(b) Some models of the axioms satisfy the additional formula

$$(4) \quad \forall x (f(x,x) = x).$$

For each of the examples (Ex1–Ex3), determine whether or not (4) holds.

(3 marks)

(c) Show that if all four formulae (1)–(4) hold in some interpretation then the formula

$$\forall x (f(x, f(x,y)) = f(y, f(y,x)))$$

must also hold.

(3 marks)

(d) Another formula which may hold for some models of the axioms is

$$(5) \quad \forall x \exists y (f(x,y) = i).$$

Again determine, stating your reasons, which of the interpretations (Ex1)–(Ex3) satisfy (5).

(3 marks)

Continued . . .

(e) By considering the expression $f(x, f(x, y))$ in two different ways, where $f(x, y) = i$, show that if (1)–(5) all hold, then the domain contains only one element.

(4 marks)

(f) State what is meant by a *complete axiomatisation* of a first-order theory. Explain why the examples in this question show that the axiom set $\{$1,2,3$\}$ cannot be a complete axiomatisation of any theory. What about the set $\{$1,2,3,4,5$\}$?

(4 marks)

5. *Essay question* Discuss arguments for and/or against *one* of the following statements:

(a) All computation consists in the transformation of strings over a finite input alphabet into strings over a finite output alphabet in such a way that the relationship between the input strings and the output strings can be expressed by means of a finite set of first-order formulae.

(b) In view of Gödel's incompleteness theorems, the view that human intelligence arises solely as a result of the computational activity of the brain cannot be upheld.

(20 marks)

End of Paper

APPENDIX

Rewriting rules for the truth-tree procedure:

1. **Non-branching rules.**

Given	Add to branch	
$P \wedge Q$	P, Q	
$\neg(P \vee Q)$	$\neg P, \neg Q$	
$\neg(P \rightarrow Q)$	$P, \neg Q$	
$\neg\neg P$	P	
$\forall x \Phi(x)$	$\Phi(t)$	where t is any term constructed from constants occurring in the current branch
$\neg\exists x \Phi(x)$	$\neg\Phi(t)$	
$\exists x \Phi(x)$	$\Phi(a)$	where a is a constant which does not appear in the current branch
$\neg\forall x \Phi(x)$	$\neg\Phi(a)$	

2. **Branching rules.**

Given	Add to left branch	Add to right branch
$P \vee Q$	P	Q
$P \rightarrow Q$	$\neg P$	Q
$P \leftrightarrow Q$	P, Q	$\neg P, \neg Q$
$\neg(P \wedge Q)$	$\neg P$	$\neg Q$
$\neg(P \leftrightarrow Q)$	$P, \neg Q$	$\neg P, Q$