COM3312

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING AND COMPUTER SCIENCE
Department of Computer Science

BSc Mathematics with Computer Science, Year 3
BSc Cognitive Science, Year 3 (option)
BSc Computer Science, Year 3 (option)
BSc Computer Science with Management Science, Year 3 (option)
BSc Computer Science with Mathematics, Year 3 (option)
BSc Media Computing, Year 3 (option)

Logic and Computation

TWO HOURS

Answer THREE out of the five questions.

Candidates are advised to spend FORTY MINUTES on each question.

No electronic calculators of any sort are to be used during the course of this examination.

1. *Models and interpretations*

(a) Explain the meanings of the following pairs of terms as applied to first-order logic, taking particular care to clarify the distinction between the two members of each pair:

(i) *interpretation* and *model*
(ii) *logical* and *non-logical* vocabulary
(iii) *valid* and *invalid* inference

(6 marks)

In the remainder of this question we shall work with a first-order language $\mathcal{L}$ containing a one-place predicate letter P and a two-place predicate letter R, and no other non-logical vocabulary. In this language we shall consider the following closed formulae:

A: $\forall x \forall y((P(x) \wedge R(x, y)) \rightarrow P(y))$
B: $\forall x(\neg P(x) \rightarrow \exists y(R(x, y) \wedge \neg P(y)))$
C: $\forall x(\exists y(R(x, y) \wedge \neg P(y)) \rightarrow \neg P(x))$

(b) Verify that all three of these formulae are satisfied by each of the following interpretations:

Interpretation 1:	Domain is the set $\mathbb{N}$ of all natural numbers
	$P(x)$ means "x is even"
	$R(x, y)$ means "x is a proper factor of y"
Interpretation 2:	Domain is the power set of $\mathbb{N}$
	$P(x)$ means "x is infinite"
	$R(x, y)$ means "x is a proper subset of y"

(6 marks)

(c) Of the two formulae B and C, one is a logical consequence of A, and the other is not. Determine which is which, and justify your answer by

(i) explaining why the formula which you claim to be a logical consequence of A really does follow from A; and
(ii) showing that the other formula is not a consequence of A by exhibiting a suitable interpretation of the language $\mathcal{L}$.

(8 marks)

2. *Proof procedures.*

(a) What is meant by a *proof procedure* for a logic?

(2 marks)

(b) Explain the meanings of the terms *decidable*, *semi-decidable*, and *undecidable* as applied to a logical system.

(3 marks)

The *truth tree* proof procedure makes use of a fixed set of rewriting rules by which formulae are replaced or supplemented by new formulae constructed from them in accordance with the rules. There are two kinds of rule: branching rules and non-branching rules. The complete set of rules for first-order logic is listed in the appendix to this question paper.

(c) A non-branching rule has the form "Given a formula F in a branch of the tree, add formulae $F_1, \ldots, F_n$ to that branch". What logical relationship between F and the formulae $F_1, \ldots, F_n$ must hold in order for rules of this kind to be justified?

Give a similar explanation for the branching rules.

(4 marks)

(d) Use the truth tree method to establish the validity or invalidity of the following inference, explaining carefully how the truth tree justifies the conclusion that you draw.

$$\frac{\begin{array}{c}\forall x(P(x) \to Q(x)) \\ \exists x(P(x) \wedge R(x))\end{array}}{\exists x(Q(x) \wedge R(x))}$$

(7 marks)

(e) Sometimes the truth tree procedure will fail to terminate. What can be said about the initial set of formulae in such a case? Could the truth tree method be improved so that this phenomenon does not occur? If so, how? If not, why not?

(4 marks)

Please Turn Over

3. *First-order theories.*

(a) What is meant by a *first-order theory*?

(2 marks)

(b) Given a domain D on which certain predicates and functions are defined, what is meant by *the* first-order theory of D? Explain why the first order theory of such a domain must *be* a first-order theory in the sense you explained in part (a).

(3 marks)

Consider the domain H in which the individuals are human beings, and the following predicates are defined:

$F(x)$	"x is female"
$P(x, y)$	"x is a parent of y"

Assuming as an idealisation that the lineage of ancestors of any individual extends indefinitely far into the past, the first-order theory of this domain will include the formula

$$\forall x \exists y (F(y) \wedge P(y, x)).$$

(c) Express in ordinary English the meaning of this formula under the interpretation suggested above.

(1 mark)

(d) Write down six more formulae which are true of this domain under the assumed idealisation. [**Hint**: You might consider such facts as that everyone has a father, that no-one has more than two parents, that no-one is their own parent, and so on. The available logical vocabulary may be taken to include the identity symbol "$=$" in addition to the usual connectives and quantifiers.]

(6 marks)

(e) The formulae you wrote down in (d) could be considered to be (part of) an *axiomatisation* of the first-order theory of the domain H. This axiomatisation may or may not be *sound* and/or *complete*. Explain the meanings of these three italicised terms.

(4 marks)

Continued . . .

(f) What modifications would you make to the formulae you wrote down in (d) if you were required to assume the alternative idealisation that all humans are descended from a single ancestral pair of humans (“Adam” and “Eve”) who did not themselves have human parents. (You may introduce individual constants into the language to handle this case.)

(4 marks)

4. *Recursive and recursively enumerable languages.*

(a) Define the terms *recursive* and *recursively enumerable* as applied to formal languages.

(4 marks)

(b) Show that a recursive language is necessarily recursively enumerable, and give an example of a recursively enumerable language which is not recursive (you do not need to prove that it is not).

(4 marks)

(c) If X is a recursive language and Y is a recursively enumerable language, determine which of the following statements are true, explaining your reasons:

(i) $X \cap Y$ is recursive.

(ii) $X \cup Y$ is recursively enumerable.

(iii) $X \setminus Y$ is recursively enumerable.

(6 marks)

(d) The well-known Halting Problem for Turing Machines can be characterised by the fact that a certain formal language is not recursively enumerable. Define a suitable language, and explain how the unsolvability of the Halting Problem follows from the fact that it is not recursively enumerable.

(6 marks)

5. *Essay question.*
Write an essay on *one* of the following topics:

(a) How Turing showed that the *Entscheidungsproblem* could not be solved.

(b) Gödel’s incompleteness theorem.

(20 marks)

End of Paper

APPENDIX

Rewriting rules for the truth-tree procedure:

1. **Non-branching rules.**

Given	Add to branch	
$P \wedge Q$	P, Q	
$\neg(P \vee Q)$	$\neg P, \neg Q$	
$\neg(P \rightarrow Q)$	$P, \neg Q$	
$\neg\neg P$	P	
$\forall x \Phi(x)$	$\Phi(t)$	where t is any term constructed from constants occurring in the current branch
$\neg\exists x \Phi(x)$	$\neg\Phi(t)$	
$\exists x \Phi(x)$	$\Phi(a)$	where a is a constant which does not appear in the current branch
$\neg\forall x \Phi(x)$	$\neg\Phi(a)$	

2. **Branching rules.**

Given	Add to left branch	Add to right branch
$P \vee Q$	P	Q
$P \rightarrow Q$	$\neg P$	Q
$P \leftrightarrow Q$	P, Q	$\neg P, \neg Q$
$\neg(P \wedge Q)$	$\neg P$	$\neg Q$
$\neg(P \leftrightarrow Q)$	$P, \neg Q$	$\neg P, Q$