**COM3412**

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING AND COMPUTER SCIENCE
Department of Computer Science

B.Sc. Computer Science, Year 3 (option)
B.Sc. Computer Science with European Study, Year 4 (option)
B.Sc. Cognitive Science, Year 3 (option)
B.Sc. Mathematics with Computer Science, Year 3 (option)

# *Logic and Computation*

TWO HOURS

Answer THREE out of the five questions.

Candidates are advised to spend FORTY MINUTES on each question.

No electronic calculators of any sort are to be used during the course of this examination.

**1.** (a) Explain what is meant by

(i) The Halting Problem for Turing Machines,

(ii) The Decision Problem for First-order Logic,

and discuss the practical importance of these problems.

**(8 marks)**

(b) Outline the steps by which it can be shown that these two problems are equivalent.

**(8 marks)**

(c) What is our current state of knowledge regarding the solvability or otherwise of these problems?

**(4 marks)**

**2.** (a) Explain what is meant by the first-order theory of arithmetic for the natural numbers.

**(4 marks)**

(b) Gödel proved that the first-order theory of arithmetic cannot be finitely axiomatised. Explain what this means, and outline the method by which he proved it.

**(6 marks)**

(c) Peano Arithmetic is the First-order Theory defined by the axioms:

**(S1)** $\forall x \neg (sx = 0)$
**(S2)** $\forall x \forall y (sx = sy \rightarrow x = y)$
**(A1)** $\forall x (x + 0 = x)$
**(A2)** $\forall x \forall y (x + sy = s(x + y))$
**(M1)** $\forall x (x * 0 = 0)$
**(M2)** $\forall x \forall y (x * sy = (x * y) + x)$

together with the axiom schema

**(Ind)** $\Phi(0) \wedge \forall x (\Phi(x) \rightarrow \Phi(x+1)) \rightarrow \forall x (\Phi(x))$

Show that the formula $\forall x (s0 * x = x)$ is a theorem of Peano Arithmetic.

**(7 marks)**

(d) What can be said about the relation between Peano Arithmetic and the first-order theory of arithmetic in the light of Gödel's proof?

**(3 marks)**

**3.** The first-order theory of parts and wholes uses a non-logical vocabulary consisting of two binary predicates $P$ and $O$ (for 'part of' and 'overlap' respectively). The predicate $O$ is defined to be reflexive, symmetric, and extensional, using the following axioms:

**(OR)** $\forall x O(x, x)$
**(OS)** $\forall x \forall y (O(x, y) \rightarrow O(y, x))$
**(OE)** $\forall x \forall y (\forall z (O(x, z) \leftrightarrow O(y, z)) \rightarrow x = y)$

The predicate $P$ is defined in terms of $O$ by the rule

**(PO)** $\forall x \forall y (P(x, y) \leftrightarrow \forall z (O(z, x) \rightarrow O(z, y)))$

(a) Explain in words and/or diagrams the meaning of the axioms (OE) and (PO).

**(2 marks)**

(b) Use the axioms to prove that $P$ is reflexive, transitive, and antisymmetric:

**(PR)** $\forall x P(x, x)$
**(PT)** $\forall x \forall y \forall z (P(x, y) \wedge P(y, z) \rightarrow P(x, z))$
**(PA)** $\forall x \forall y (P(x, y) \wedge P(y, x) \rightarrow x = y)$

Note: Fully formal proofs are not required.

**(6 marks)**

An interpretation for this language is defined as follows. The domain is the set of all non-empty sets of integers (i.e., $\mathcal{P}(\mathbb{Z}) \setminus \{\emptyset\}$), and $O$ is interpreted so that $O(x, y)$ is true if and only if $I(x) \cap I(y) \neq \emptyset$ (where $I(x)$ means the set denoted by '$x$').

(c) Verify that the axioms OR, OS, and OE are all satisfied by this interpretation. How must $P$ be interpreted so that PO is also satisfied?

**(9 marks)**

(d) Explain why the formula

**(OT)** $\forall x \forall y \forall z (O(x, y) \wedge O(y, z) \rightarrow O(x, z))$

is *not* a logical consequence of the axioms OR, OS, and OE.

**(3 marks)**

**Please Turn Over**

**4.** The standard definition of $\models$ for first-order logic is that $\Sigma \models C$ if and only if every interpretation which satisfies all the formulae in $\Sigma$ also satisfies $C$.

(a) Show that, with $\models$ defined in this way, any inference with contradictory premisses must be valid. What property of the *conclusion* can similarly guarantee that any inference in which the conclusion has that property must be valid?

**(4 marks)**

A logical consequence relation $\models$ is *monotonic* if, for any formula $C$ and sets of formulae $\Sigma, \Sigma'$, if $\Sigma \models C$ and $\Sigma \subseteq \Sigma'$ then $\Sigma' \models C$.

(b) Show that the standard logical consequence relation, as defined above, is monotonic.

**(5 marks)**

(c) The monotonicity of $\models$ implies that if the inference

$$P_1, \ldots, P_n, \text{ therefore } C$$

is valid, so is the inference

$$P_1, \ldots, P_n, \neg C, \text{ therefore } C.$$

On the face of it, this may seem paradoxical. Explain carefully what is going on here.

**(4 marks)**

(d) It is sometimes claimed that classical logic is an inappropriate tool for modelling everyday reasoning, on the grounds that the latter is *non-monotonic*. Give an example to illustrate this claim, and discuss whether it is in fact justified.

**(7 marks)**

**5. Answer ONE of the following questions:**

(a) Discuss the meanings of the terms 'procedure', 'algorithm', 'heuristic', and 'computational task', explaining the relationships between them. To what extent are these concepts applicable to (i) everyday computing applications, and (ii) the tasks performed by human beings in everyday life?

**(20 marks)**

(b) Define clearly the meanings of the terms 'P', 'NP', and 'NP-complete'. State Cook's Theorem, and describe in outline how it may be proved. Explain the key role of Cook's Theorem in developing the theory of NP-completeness, and discuss the implications of this theory for practical computation.

**(20 marks)**

**End of Paper**