COM3412

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING AND COMPUTER SCIENCE
Department of Computer Science

Logic and Computation

TWO HOURS

Answer THREE out of the five questions.
All questions carry equal weight.

Candidates are advised to spend FORTY MINUTES on each question.

Calculators may be used.

1. (a) Explain what is meant by the terms *interpretation*, *model*, and *logical consequence* in formal logic.

(3 marks)

Consider the following first-order formulae:

$$\begin{array}{rl} P: & \forall x \forall y \exists z A(x,y,z) \\ Q: & \forall x \forall y \exists z A(x,z,y) \\ R: & \forall x \forall y \forall z (A(x,y,z) \rightarrow A(y,x,z)) \\ S: & \exists x \forall y A(x,y,y) \end{array}$$

and the interpretation $\mathcal{I}_1$ with domain $\mathbb{Z}$ (i.e., the set of all integers), in which $A(x,y,z)$ is interpreted to mean that z is the sum of x and y.

(b) Explain informally why interpretation $\mathcal{I}_1$ is a model for $\{P, Q, R, S\}$.

(4 marks)

(c) Interpretation $\mathcal{I}_2$ is the same as $\mathcal{I}_1$ except that its domain is $\mathbb{N}$. Explain why $\mathcal{I}_2$ is not a model for $\{P, Q, R, S\}$.

(4 marks)

(d) Find an interpretation $\mathcal{I}_3$ for $\{P, \neg Q, R, \neg S\}$.

(4 marks)

(e) By attempting to formalise it, expose the fallacy in the following informal argument:

> "From Q, with $x = y$, it follows that for every x there is a z such that $A(x,z,x)$. By R, this means that for every x there is a z such that $A(z,x,x)$. Replacing x by y and z by x, it follows that there is an x such that for every y, $A(x,y,y)$. Hence S is a logical consequence of Q and R."

(5 marks)

2. The first-order theory of the ordering on the real numbers satisfies the following axioms:

1. $\forall x \neg(x < x)$
2. $\forall x \forall y \forall z(x < y \wedge y < z \rightarrow x < z)$
3. $\forall x \forall y(x < y \vee x = y \vee y < x)$
4. $\forall x \exists y(y < x)$
5. $\forall x \exists y(x < y)$
6. $\forall x \forall y(x < y \rightarrow \exists z(x < z \wedge z < y))$

(a) Explain what is meant by saying that axioms 1–6 *completely axiomatise* the theory.

(2 marks)

(b) Explain what each of the axioms means and what property of the ordering it expresses.

(3 marks)

(c) Show that the axioms logically imply the formula

$$\forall w \forall x \forall y \forall z(w < y \wedge x < z \rightarrow w < z \vee x < y).$$

(A fully formal proof is not required, but would be acceptable.)

(3 marks)

(d) Show that any non-empty model for axioms 1–6 must have an infinite domain. Identify *two* minimal subsets of 1–6 which still have this property.

(5 marks)

(e) Verify that all but one of the axioms are satisfied by the ordering of the *integers*, and suggest a suitable replacement for the axiom which is not satisfied.

(5 marks)

(f) What can be said about the first-order theory of the ordering of the rational numbers?

(2 marks)

Please Turn Over

3. (a) Consider the following problem: Given a finite set of integers S and an integer n, determine whether or not S contains a subset whose members sum to n. Explain what is meant by the complexity class NP, and show that this problem belongs to that class.

(4 marks)

(b) A subclass of NP is the class NP-complete. How is this class defined? Give an example of one problem in NP which is also in NP-complete, and one problem in NP which is *not* in NP-complete.

(3 marks)

(c) The first problem to be identified as NP-complete was SAT, the Satisfiability Problem for the Propositional Calculus. Explain what this problem is, and outline the method by which it was shown to be NP-complete.

(7 marks)

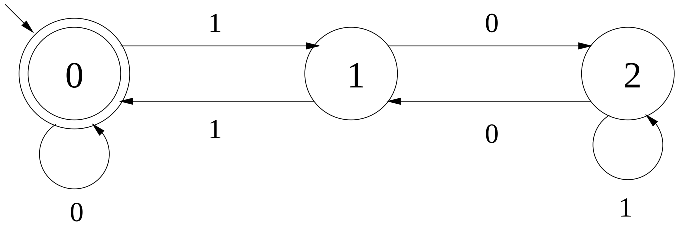
(d) Explain how the NP-completeness of SAT can be used to establish the existence of other NP-complete problems.

(4 marks)

(e) Discuss the implications of NP-completeness for practical computing concerns.

(2 marks)

4. (a) Consider the Turing machine with input alphabet $\{1\}$ specified by the following transition diagram.



Describe the behaviour of this machine when it is run with each of the following initial conditions:

(i) A blank tape.

(ii) A tape containing a single '1' in some position but otherwise blank.

What is the *precise* condition that the machine eventually reaches the halt state? Prove that this machine must either reach the halt state or run for ever.

(6 marks)

(b) What is meant by the Halting Problem for Turing machines? Show that the Halting Problem for Turing machines cannot be solved *by a Turing Machine*.

(6 marks)

(c) What reasons are there for supposing that the Halting Problem for Turing machines might or might not be solvable by means other than a Turing machine?

(4 marks)

(d) What are the implications of the Halting Problem for Turing machines for the Decision Problem for First-order Logic?

(4 marks)

Please Turn Over

5. *Answer ONE of the following:*

(a) First-order logic is both *compact* and *monotonic*. Explain what is meant by these properties, and discuss their implications in the context of the expressive power and usability of the logic.

(20 marks)

(b) Gödel showed that the first-order theory of Natural Number arithmetic is not completely axiomatisable. Explain what this means and outline the method by which the proof was accomplished. A number of prominent researchers have claimed that Gödel's theorem has implications outside the world of pure mathematics and logic; do you think these claims can be upheld?

(20 marks)

End of Paper