COM3412

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING, COMPUTER SCIENCE AND MATHEMATICS
Department of Computer Science

Logic and Computation

TWO HOURS

Answer question 1, and two out of the four other questions.
Question 1 is worth 40 marks. Other questions are worth 30 marks each.

Candidates are advised to spend FORTY-FIVE minutes on question 1
and THIRTY-FIVE minutes on other questions.

No electronic calculators of any sort are to be used during the course of this examination.

Compulsory question

1. (a) Explain what is meant by

- a first-order language;
- an interpretation for a first-order language;
- a model for a set of first-order formulae.

How is the logical notion of *validity* defined in terms of these concepts?

(13 marks)

(b) The 'or'-elimination rule states that if a conclusion ψ can be validly inferred both from the premiss-set $\Sigma \cup \{\phi_1\}$ and from the premiss-set $\Sigma \cup \{\phi_2\}$, then it can also be validly inferred from the premiss-set $\Sigma \cup \{\phi_1 \vee \phi_2\}$.

- State the semantic rule for disjunction.
- Use the definition of validity given in (a) and the semantic rule for disjunction to justify the 'or'-elimination rule.

(12 marks)

(c) Similarly state and justify the 'or'-introduction rule.

(5 marks)

(d) Show how the 'or'-introduction and 'or'-elimination rules can be used to derive $(P \vee Q) \vee R$ from $P \vee (Q \vee R)$.

(10 marks)

(Total 40 marks)

2. (a) Explain what is meant by describing a logic as *compact*.

(4 marks)

(b) Show that the existence of a sound and complete proof system for a logic is sufficient to ensure that the logic is compact.

(10 marks)

(c) In a first-order language (with identity) which contains a single one-place predicate P, write down formulae which say

- At least one object is P.
- At least two objects are P.
- At least three objects are P.

Use the compactness of first-order logic to show that there is no formula in this language which says that infinitely many objects are P.

(16 marks)

(Total 30 marks)

3. (a) Peano Arithmetic is the first-order theory defined by the axioms

S1. $\forall x \neg(sx = 0)$

S2. $\forall x \forall y(sx = sy \to x = y)$

A1. $\forall x(x + 0 = x)$

A2. $\forall x \forall y(x + sy = s(x + y))$

M1. $\forall x(x * 0 = 0)$

M2. $\forall x \forall y(x * sy = (x * y) + x)$

together with the axiom schema

Ind. $\Phi(0) \wedge \forall x(\Phi(x) \to \Phi(sx)) \to \forall x(\Phi(x))$.

Show that the following formulae are theorems of Peano Arithmetic:

(i) $\forall x \forall y \forall z(x + (y + z) = (x + y) + z)$.

(ii) $\forall x \forall y \forall z(x + z = y + z \to x = y)$.

Make sure that every step in your reasoning is made explicit.

(16 marks)

(b) Explain what is meant by the first-order theory of arithmetic for the natural numbers, and explain briefly how we know that this theory is not finitely axiomatisable. What is the relationship between this theory and Peano Arithmetic?

(14 marks)

(Total 30 marks)

4. (a) Study the Turing machine whose transition diagram is shown below, and answer the following questions about it:

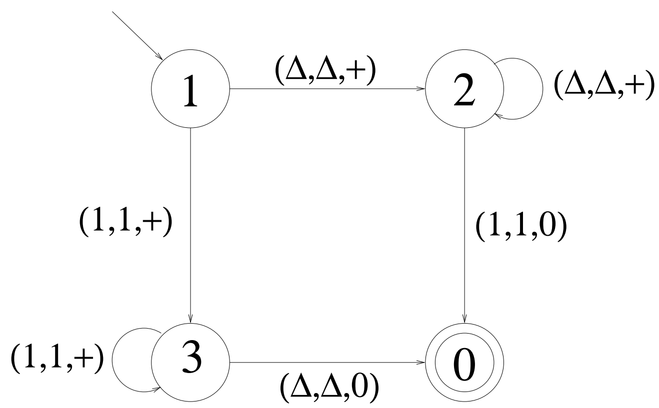
(i) Give examples of

- an input which results in the machine reaching the halt state;
- an input which causes the machine to continue running for ever.

(ii) With this machine, no input over the alphabet $\{\Delta, 1\}$ can cause the machine to halt in a non-halting state.

- Justify this statement.
- Modify the machine by removing just one transition so that it *can* halt in a non-halting state.
- Specify an input to your modified machine for which this behaviour will occur.

(12 marks)



(b) There does not exist an effective procedure which will enable one to determine, for an arbitrary Turing machine and arbitrary input tape, whether or not the Turing machine will eventually halt when run with that tape as input.

(i) Explain how Turing established the truth of this assertion.

(ii) Outline the method by which he used this fact to prove that First-order Logic is undecidable.

(18 marks)

(Total 30 marks)

Please Turn Over

5. *Essay question*

Write an essay on one of the following topics

(a) Cook's Theorem and the theory of NP-completeness.

(b) The Church-Turing Thesis.

(Total 30 marks)

End of Paper