COM3412

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING, COMPUTER SCIENCE
AND MATHEMATICS

Logic and Computation

TWO HOURS

Answer question 1, and two out of the four other questions.
Question 1 is worth 40 marks. Other questions are worth 30 marks each.

Candidates are advised to spend FORTY-FIVE minutes on question 1
and THIRTY-FIVE minutes on other questions.

No electronic calculators of any sort are to be used during the course of this examination.

COM3412

Compulsory question

1. (a) Explain what is meant by a *logical inference*, and what it means to say that an inference is *valid*.

(5 marks)

(b) Which of the following inferences are valid in ordinary Propositional Calculus? Give reasons for your answers.

$$\frac{A}{A \wedge B} \qquad \frac{A \wedge B}{A} \qquad \frac{A}{A \vee B} \qquad \frac{A \vee B}{A}$$

(8 marks)

(c) What is meant by a *proof system* for a logic? In this context, explain the meanings of the terms 'sound', 'complete', and 'decision procedure'.

(11 marks)

Consider the following "toy" logic, which resembles the Propositional Calculus (PC) except that it has only one connective, which we write '$\circ$'. The well-formed formulae are defined recursively by the rules:

1. Any schematic letter is a formula.
2. If α and β are formulae then so is $(\alpha \circ \beta)$.

Consider two rather simple-minded proof systems for this logic:

PS1 An inference is deemed valid if and only if every letter appearing in the conclusion appears in at least one of the premisses.

PS2 An inference is deemed valid if and only if every letter appearing in at least one of the premisses also appears in the conclusion.

Consider further two possible interpretations of the connective '$\circ$':

I1 $\alpha \circ \beta$ is interpreted to mean the same as the PC formula $\alpha \wedge \beta$.

I2 $\alpha \circ \beta$ is interpreted to mean the same as the PC formula $\alpha \vee \beta$.

(d) For each of the interpretations I1 and I2, discuss whether each of the proof systems PS1 and PS2 is (a) sound, and (b) complete.

(16 marks)

(Total 40 marks)

2. The standard definition of $\models$ for first-order logic is that $\Sigma \models C$ if and only if every interpretation which satisfies all the formulae in Σ also satisfies C.

(a) Show that, with $\models$ defined in this way, any inference with contradictory premisses must be valid. What property of the *conclusion* can similarly guarantee that any inference in which the conclusion has that property must be valid?

(6 marks)

A logical consequence relation $\models$ is *monotonic* if, for any formula C and sets of formulae Σ, Σ', if $\Sigma \models C$ and $\Sigma \subseteq \Sigma'$ then $\Sigma' \models C$.

(b) Show that the standard logical consequence relation, as defined above, is monotonic.

(7 marks)

(c) The monotonicity of $\models$ implies that if the inference

$$P_1, \ldots, P_n, \text{ therefore } C$$

is valid, so is the inference

$$P_1, \ldots, P_n, \neg C, \text{ therefore } C.$$

On the face of it, this may seem paradoxical. Explain carefully what is going on here.

(7 marks)

(d) It is sometimes claimed that classical logic is an inappropriate tool for modelling everyday reasoning, on the grounds that the latter is *non-monotonic*. Give an example to illustrate this claim, and discuss whether it is in fact justified.

(10 marks)

(Total 30 marks)

Please Turn Over

3. (a) What is meant by a *first-order theory*?

(3 marks)

(b) Given a domain D on which certain predicates and functions are defined, what is meant by *the* first-order theory of D? Explain why the first order theory of such a domain must *be* a first-order theory in the sense you explained in part (a).

(5 marks)

Consider the domain H in which the individuals are human beings, and the following predicates are defined:

$F(x)$ "x is female"
$P(x, y)$ "x is a parent of y"

Assuming as an idealisation that the lineage of ancestors of any individual extends indefinitely far into the past, the first-order theory of this domain will include the formula

$$\forall x \exists y (F(y) \wedge P(y, x)).$$

(c) Express in ordinary English the meaning of this formula under the interpretation suggested above.

(2 marks)

(d) Write down six more formulae which are true of this domain under the assumed idealisation. [**Hint**: You might consider such facts as that everyone has a father, that no-one has more than two parents, that no-one is their own parent, and so on. The available logical vocabulary may be taken to include the identity symbol "$=$" in addition to the usual connectives and quantifiers.]

(9 marks)

(e) The formulae you wrote down in (d) could be considered to be (part of) an *axiomatisation* of the first-order theory of the domain H. This axiomatisation may or may not be *sound* and/or *complete*. Explain the meanings of these three italicised terms in this context.

(6 marks)

Continued . . .

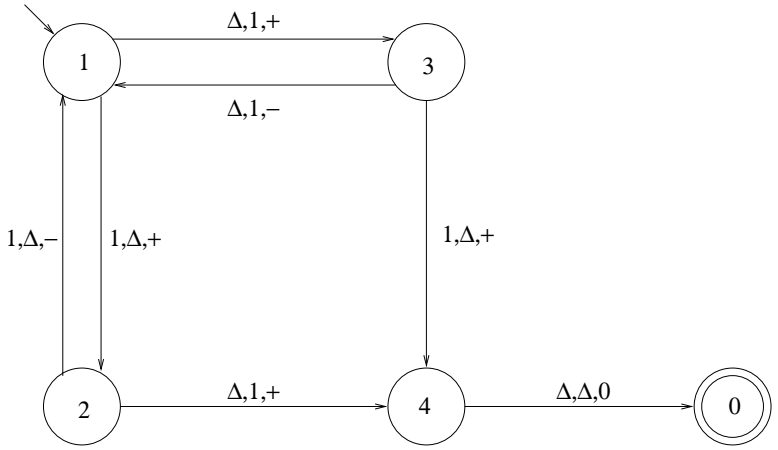
(f) What modifications would you make to the formulae you wrote down in (d) if you were required to assume the alternative idealisation that all humans are descended from a single ancestral pair of humans ("Adam" and "Eve") who did not themselves have human parents. (You may introduce individual constants into the language to handle this case.)

(5 marks)

(Total 30 marks)

Please Turn Over

4. (a) Consider the Turing Machine specified by the following transition diagram:



For this Turing Machine, identify an example of each of the following:

i. An initial tape for which the machine, when run, fails to terminate.

ii. An initial tape for which the machine, when run, eventually reaches the halt state.

iii. An initial tape for which the machine, when run, becomes stuck in a non-halting state.

Justify your answers by reference to the sequence of configurations assumed by the machine when run with each of the initial tapes you have identified.

(12 marks)

(b) Consider the statements

1. There is no effective procedure for determining whether or not an arbitrary Turing Machine, when run with an arbitrary initial tape, will eventually reach the halt state.
2. There is no effective procedure for determining whether or not an arbitrary inference in first-order predicate calculus is valid.

Explain the connection between these statements, and outline the reasons we have for believing them both to be true.

(18 marks)

(Total 30 marks)

5. Discuss the contributions made to the theory of logic and computation by **either** Kurt Gödel **or** Stephen Cook.

(Total 30 marks)

End of Paper