# COM3412R (2002): Specimen Solutions

1. (a) An interpretation must specify a domain $\Delta$ (which may be any set), and it must map each element of the non-logical vocabulary of the language onto an element related to $\Delta$, as follows:

   an individual constant is mapped onto an element of $\Delta$;
   an $n$-ary function symbol is mapped onto a function from $\Delta^n$ to $\Delta$;
   an $n$-ary predicate symbol is mapped onto a subset of $\Delta^n$.

   There are general rules for determining the truth value of each sentence in the language relative to a given interpretation. A model for a set of formulae is any interpretation under which each formula in the set is true.

   (b) Under $\mathcal{I}_1$:

   $A$ says that no number is less than itself;
   $B$ says that the 'less than' relation is transitive;
   $C$ says that, of any two distinct real numbers, one is less than the other.

   All these are obviously true.

   (c) One possibility for $\mathcal{I}_2$ is as follows:

   The domain is the power set of some set $U$.
   $P$ is interpreted to mean 'is a proper subset of'.

   Then $A$ is satisfied since no set is a proper subset of itself.
   $B$ is satisfied since the proper subset relation is transitive.
   $C$ is not satisfied, since, e.g., neither of $\{1, 2\}, \{1, 3\}$ is a subset of the other.
   For $C$ to be a logical consequence of $\{A, B\}$ it is necessary that every model for $\{A, B\}$ satisfies $C$. So the existence of model $\mathcal{I}_2$ for $\{A, B\}$ which does not satisfy $C$ proves that $C$ is not a logical consequence of $\{A, B\}$.

   (d) We need to find an interpretation which satisfies $B$ and $C$ but not $A$. One possibility is:

   The domain is the set of all real numbers [as in $\mathcal{I}_1$];
   $P$ is interpreted to mean 'is less than or equal to'.

   Then $B$ and $C$ are still satisfied, but $A$ is not, since any number is less than or equal to itself.
   Hence $A$ is not a logical consequence of $\{B, C\}$.

   (e) Assume $A$ and $B$.
   Suppose $P(x, y) \wedge P(y, x)$. Then by $B$, we have $P(x, x)$, contradicting $A$. Hence we cannot have both $P(x, y)$ and $P(y, x)$.
   Now suppose $P(x, y) \wedge x = y$. Then again we have $P(x, x)$, contradicting $A$, so we cannot have both $P(x, y)$ and $x = y$.
   Similarly, we cannot have both $P(y, x)$ and $x = y$.
   Hence we can have at most one of $P(x, y)$, $P(y, x)$, and $x = y$.

2. (a) $\phi$ is uniquely recoverable from $g(\phi)$ because each positive integer has a unique factorisation into primes.
   $2430 = 2^1 \times 3^5 \times 5^1$ which is the Gödel number of the formula '$0 = 0$'.

   (b) A formula saying that $1 + 1 = 2$ is '$s0 + s0 = ss0$'. It's Gödel number is

$$2^3 \times 3^1 \times 5^{17} \times 7^3 \times 11^1 \times 13^5 \times 17^3 \times 19^3 \times 23^1.$$

(c) If the number is $2^n k$, where $k$ is odd, then $k$ is the Gödel number of the first symbol in the string. If it is divisible by 32, then $n \geq 5$. But if $n = 5$, the first symbol is '$=$' so it is not a well-formed formula. Hence $n \geq 7$, so the number is divisible by $s^7 = 128$.
If $g(\phi)/128$ is odd, then $n = 7$, so the first symbol is '$\neg$'. Hence *the formula is a negation*.

(d) For any formula $\phi$, $g(\phi) = 2^n k$, where $n \geq 1$, so $g(\phi)$ is even. Hence for the Gödel number of a sequence, every exponent is even, which means that the number is a square. This cannot happen in the case of the Gödel number of a formula, in which every exponent is odd.

(e) Bookwork. The main points are that Gödel showed that many of the syntactic properties of formulae and sequences of formulae corresponded to arithmetical properties of their Gödel numbers. This meant that formulae expressing arithmetical properties of Gödel numbers could be interpreted as also expressing syntactical properties of formulae or sequences of formulae; Gödel showed how to exploit this by setting up a formula which, under the second interpretation, states that it itself is not provable from the assumed axiomatisation. This formula must then be either true but not provable (making the axiomatisation incomplete) or provable but not true (making it unsound).

3. (a) In Ex1, $i$ must be 0, since for real numbers, $x + y = x$ if and only if $y = 0$.
In Ex2, $i$ must be 1, since for real numbers, $x.y = x$ if and only if $y = 1$.
In Ex3, $i$ must be $\emptyset$, since $\emptyset$ is the only set $y$ for which we *always* have $x \cup y = x$.

(b) In (Ex1), (4) is not satisfied since, e.g., $2 + 2 \neq 2$.
In (Ex2), (4) is not satisfied since, e.g., $2 \times 2 \neq 2$.
In (Ex3), (4) is satisfied, since for any set $X$ we have $X \cup X = X$.

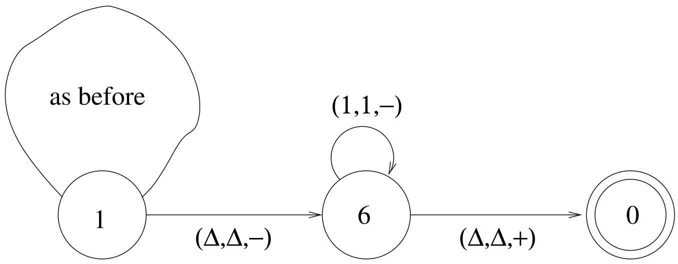
(c)
$$\begin{array}{rcll} f(x, f(x, y)) & = & f(f(x, x), y) & \text{(by 2)} \\ & = & f(x, y) & \text{(by 4)} \\ & = & f(y, x) & \text{(by 1)} \\ & = & f(f(y, y), x) & \text{(by 4)} \\ & = & f(y, f(y, x)) & \text{(by 2)} \end{array}$$

(d) In (Ex1), (5) is satisfied since for any $x$ we have $x + (-x) = 0$.
In (Ex2), (5) is not satisfied, since there is no $y$ such that $0 \times y = 1$.
In (Ex3), (5) is not satisfied, since if $X \neq \emptyset$ then there is no set $Y$ such that $X \cup Y = \emptyset$.

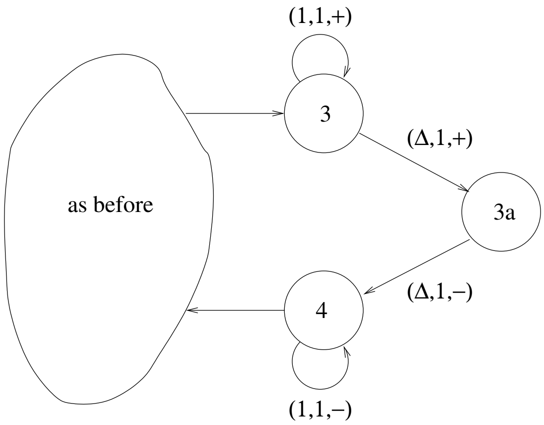
(e)
$$\begin{array}{rcll} f(x, f(x, y)) & = & f(f(x, x), y) & \text{(by 2)} \\ & = & f(x, y) & \text{(by 4)} \\ & = & i & \text{(by assumption)} \\ f(x, f(x, y)) & = & f(x, i) & \text{(by assumption)} \\ & = & x & \text{(by 3)} \end{array}$$
By (5), for each $x$ there is a $y$ such that $f(x, y) = i$, so by the above argument, $x = f(x, f(x, y)) = i$, so all elements are equal to $i$, i.e., the domain contains only one element.

(f) A complete axiomatisation of a first-order theory is a set of formulae whose logical consequences are precisely the formulae of the theory.
If $\{1, 2, 3\}$ were a complete axiomatisation of some theory, it would imply whichever of 4 and $\neg 4$ is in the theory. But (Ex1) satisfies $\{1, 2, 3, \neg 4\}$, while (Ex3) satisfies $\{1, 2, 3, 4\}$, so $\{1, 2, 3\}$ implies neither 4 nor $\neg 4$. [We could use 5 here instead of 4.]
The set $\{1, 2, 3, 4, 5\}$ only holds for a domain $\{i\}$ in which $f(i, i) = i$. All such domains are isomorphic, so we have a complete axiomatisation.

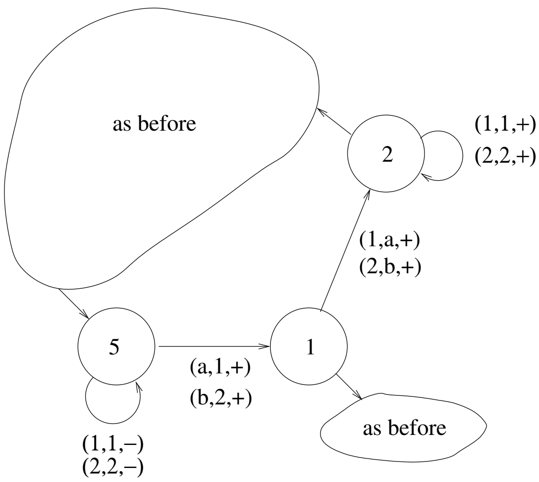
4. (a) Introduce State 6 to run back to the beginning of the string:




(b) Introduce State 3a to add an extra 1 each time round the loop:




(c) Add extra transitions at states 2 and 5 to handle the 2s:

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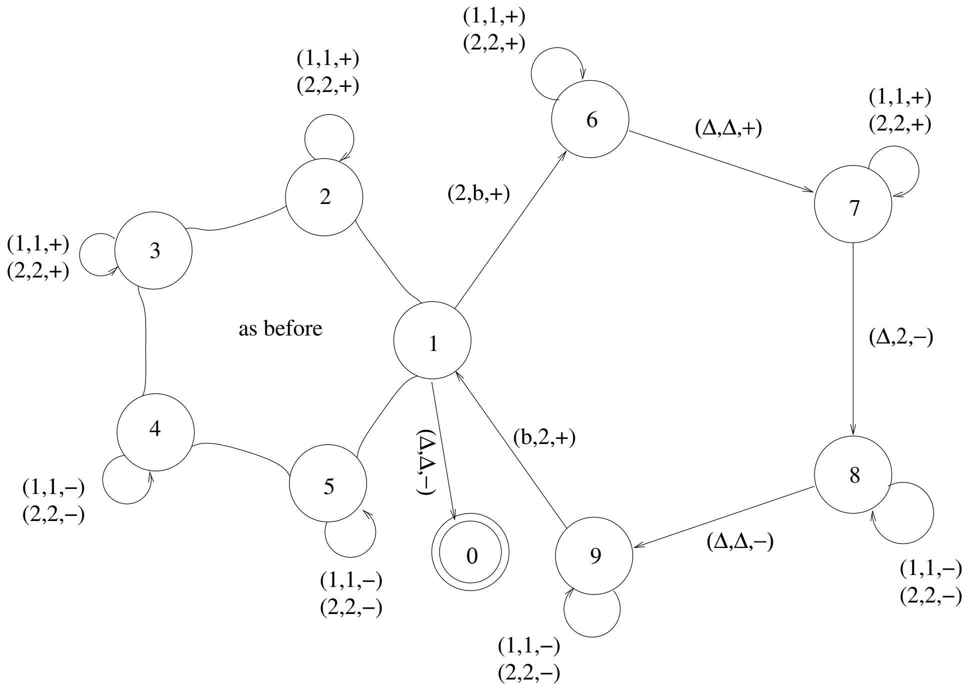
(d) Duplicate the original loop to handle the copying of 2s:




(e) As in (d), but change the transitions from 3 to 4 and from 7 to 8: