COM3412R

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING AND COMPUTER SCIENCE
Department of Computer Science

Logic and Computation

TWO HOURS

Answer THREE out of the five questions.
All questions carry equal weight.

Candidates are advised to spend FORTY MINUTES on each question.

Calculators may be used.

1. (a) Explain what is meant by an *interpretation* of a first-order language. What is the condition for an interpretation to be a *model* for a given set of formulae?

(2 marks)

The non-logical vocabulary of a language $\mathcal{L}$ consists of a single two-place predicate P. The logical vocabulary includes the identity symbol "=". In the remainder of the question we shall be interested in the following formulae of $\mathcal{L}$:

$$A: \ \forall x \neg P(x,x)$$
$$B: \ \forall x \forall y \forall z (P(x,y) \wedge P(y,z) \rightarrow P(x,z))$$
$$C: \ \forall x \forall y (P(x,y) \vee P(y,x) \vee x = y)$$

(b) An interpretation $\mathcal{I}_1$ of $\mathcal{L}$ is defined as follows: the domain is the set of all real numbers, and the predicate P denotes the relation "is less than". For this interpretation, express in English the meaning of each of the formulae A, B, and C, and hence verify that all three formulae are satisfied by the interpretation.

(3 marks)

(c) Devise an alternative interpretation $\mathcal{I}_2$ for $\mathcal{L}$ which satisfies A and B but not C. Explain why this implies that C is not a logical consequence of the set $\{A, B\}$.

(5 marks)

(d) Use the same method to show that A is not a logical conseqence of $\{B, C\}$.

(4 marks)

(e) The formula C states that for each pair of domain elements at least one of three disjuncts must hold. Show that if A and B are satisfied, then *at most* one of the three disjuncts can hold for any pair of domain elements.

(6 marks)

2. Gödel's celebrated incompleteness theorems rely for their proof on his idea of systematically assigning numbers to formulae and proofs. In one such scheme of Gödel numbering, each symbol S of the formal language is assigned an odd integer $g(S)$ as follows:

0	s	$=$	$\neg$	$\vee$	$\forall$	$($	$)$	$+$	$\times$	x	$'$
1	3	5	7	9	11	13	15	17	19	21	23

The formula ϕ consisting of symbols $S_1 S_2 \ldots S_n$ is then assigned the Gödel number

$$g(\phi) = 2^{g(S_1)} 3^{g(S_2)} 5^{g(S_3)} \cdots p_n^{g(S_n)},$$

where p_i is the ith prime number.

(a) Explain why the formula ϕ is uniquely recoverable from its Gödel number $g(\phi)$. Of which formula is 2430 the Gödel number?

(4 marks)

(b) Write down a formula which says that $1 + 1 = 2$ and express its Gödel number as a product of prime-power factors. [Do not compute the number explicitly.]

(3 marks)

(c) Explain why, if the Gödel number of a formula is divisible by 32, then it must also be divisible by 128. What do you know about a formula ϕ if you know that $g(\phi)/128$ is an odd integer?

(4 marks)

(d) The Gödel number assigned to a sequence $(\phi_1, \phi_2, \ldots, \phi_n)$ of formulae is

$$2^{g(\phi_1)} 3^{g(\phi_2)} 5^{g(\phi_3)} \cdots p_n^{g(\phi_n)}.$$

Explain why a sequence of formulae can never have the same Gödel number as a single formula.

(3 marks)

(e) Describe in outline how Gödel used a numbering system such as the one above to prove that the arithmetic of the natural numbers is not finitely axiomatisable as a first-order theory.

(6 marks)

3. The first-order theory of a commutative, associative operation f with identity i includes the following axioms:

$$\begin{array}{ll} (1) & \forall x \forall y (f(x,y) = f(y,x)) \\ (2) & \forall x \forall y \forall z (f(x, f(y,z)) = f(f(x,y),z)) \\ (3) & \forall x (f(x,i) = x) \end{array}$$

Three examples of such operations are

(Ex1) Addition on the set of real numbers.
(Ex2) Multiplication on the set of real numbers.
(Ex3) Union on the set of subsets of the real numbers.

(a) For each of the examples (Ex1)–(Ex3), state, with justification, what element is denoted by the constant symbol i appearing in the axioms.

(3 marks)

(b) Some models of the axioms satisfy the additional formula

$$(4) \quad \forall x (f(x,x) = x).$$

For each of the examples (Ex1–Ex3), determine whether or not (4) holds.

(3 marks)

(c) Show that if all four formulae (1)–(4) hold in some interpretation then the formula

$$\forall x (f(x, f(x,y)) = f(y, f(y,x)))$$

must also hold.

(3 marks)

(d) Another formula which may hold for some models of the axioms is

$$(5) \quad \forall x \exists y (f(x,y) = i).$$

Again determine, stating your reasons, which of the interpretations (Ex1)–(Ex3) satisfy (5).

(3 marks)

Continued . . .

(e) By considering the expression $f(x, f(x, y))$ in two different ways, where $f(x, y) = i$, show that if (1)–(5) all hold, then the domain contains only one element.

(4 marks)

(f) State what is meant by a *complete axiomatisation* of a first-order theory. Explain why the examples in this question show that the axiom set $\{$1,2,3$\}$ cannot be a complete axiomatisation of any theory. What about the set $\{$1,2,3,4,5$\}$?

(4 marks)

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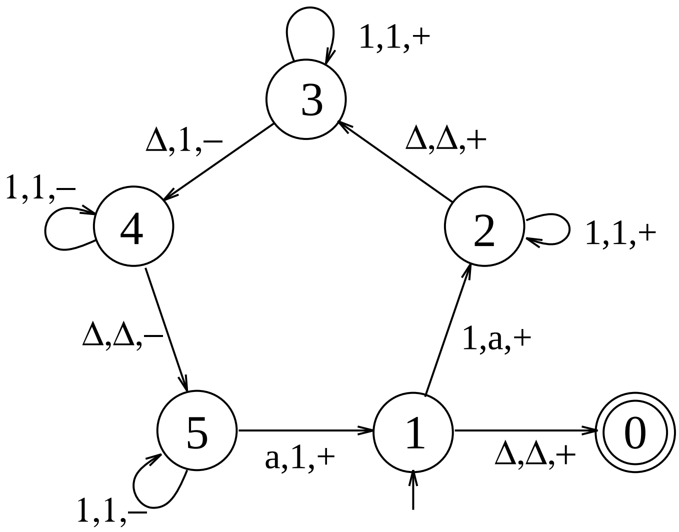
4. The diagram below shows the transition network for a Turing machine M_0.



This machine meets the following specification:

> When run with its head initially scanning the first of a string of 1s on an otherwise blank tape, M_0 copies the string to the right, separated from the original string by a single blank cell, and halts scanning the leftmost 1 of the copy:
>
> $$\boxed{1}1111 \Longrightarrow 11111\Delta\boxed{1}1111$$

In this question, you are asked to effect various modifications to this machine M_0. You can specify the modifications by drawing as much as you need of the transition diagram for the modified machine, it being understood that parts you have not shown as modified remain the same as in the illustration above.

(a) M_1 is like M_0 except that when it halts it is scanning the leftmost '1' of the original input, i.e.,

$$\boxed{1}1111 \Longrightarrow \boxed{1}1111\Delta11111$$

Construct M_1.

(3 marks)

Continued ...

(b) M_2 is like M_0 except that the copy is twice as long as the original, i.e.,

$$\boxed{1}1111 \Longrightarrow 11111\Delta\boxed{1}111111111$$

Construct M_2.

(4 marks)

(c) M_3 is like M_0 except that the string to be copied may be composed of 1s and 2s, though both 1s and 2s will appear as 1s in the 'copy':

$$\boxed{1}2212 \Longrightarrow 12212\Delta\boxed{1}1111$$

Construct M_2.

(5 marks)

(d) M_4 is like M_3 except that now the copy is identical to the original:

$$\boxed{1}2212 \Longrightarrow 12212\Delta\boxed{1}2212$$

Construct M_4.

(5 marks)

(e) M_5 is like M_4 except that now the copy is the 'negative' of the original, i.e., with 1s and 2s swapped round:

$$\boxed{1}2212 \Longrightarrow 12212\Delta\boxed{2}1121$$

Construct M_4.

(3 marks)

5. *Answer ONE of the following:*

(a) It is sometimes claimed that, in view of Gödel's incompleteness theorems, the belief that human intelligence arises solely as a result of the computational activity of the brain cannot be upheld. Discuss arguments for and against this claim.

(20 marks)

(b) Define clearly the meanings of the terms 'P', 'NP', and 'NP-complete'. State Cook's Theorem, and describe in outline how it may be proved. Explain the key role of Cook's Theorem in developing the theory of NP-completeness, and discuss the implications of this theory for practical computation.

(20 marks)

End of Paper