COM3412

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING, COMPUTING AND MATHEMATICS

Logic and Computation

TWO HOURS

Answer question 1, and two out of the four other questions.
Question 1 is worth 40 marks. Other questions are worth 30 marks each.

Candidates are advised to spend FORTY-FIVE minutes on question 1
and THIRTY-FIVE minutes on other questions.

No electronic calculators of any sort are to be used during the course of this examination.

Compulsory question

1. The first-order theory of parts and wholes uses a non-logical vocabulary consisting of two binary predicates P and O (for 'part of' and 'overlap' respectively). The predicate O is defined to be reflexive, symmetric, and extensional, using the following axioms:

(OR) $\forall x O(x,x)$
(OS) $\forall x \forall y(O(x,y) \rightarrow O(y,x))$
(OE) $\forall x \forall y(\forall z(O(x,z) \leftrightarrow O(y,z)) \rightarrow x = y)$

The predicate P is defined in terms of O by the rule

(PO) $\forall x \forall y(P(x,y) \leftrightarrow \forall z(O(z,x) \rightarrow O(z,y)))$

(a) Explain in words and/or diagrams the meaning of axioms OE and PO.

(6 marks)

(b) Use the axioms to prove that P is reflexive, transitive, and antisymmetric:

(PR) $\forall x P(x,x)$
(PT) $\forall x \forall y \forall z(P(x,y) \wedge P(y,z) \rightarrow P(x,z))$
(PA) $\forall x \forall y(P(x,y) \wedge P(y,x) \rightarrow x = y)$

Note: Fully formal proofs are not required.

(12 marks)

An interpretation for this language is defined as follows. The domain is the set of all non-empty sets of integers (i.e., $\mathcal{P}(\mathbb{Z}) \setminus \{\emptyset\}$), and O is interpreted so that $O(x,y)$ is true if and only if $I(x) \cap I(y) \neq \emptyset$ (where $I(x)$ means the set denoted by 'x').

(c) Verify that the axioms OR, OS, and OE are all satisfied by this interpretation.

(10 marks)

(d) How must P be interpreted so that PO is also satisfied? Justify your answer by showing that PO is satisfied under your suggested interpretation.

(6 marks)

Continued . . .

(e) Explain why the formula

$$\textbf{(OT)} \quad \forall x \forall y \forall z (O(x,y) \wedge O(y,z) \rightarrow O(x,z))$$

is *not* a logical consequence of the axioms OR, OS, and OE.

(6 marks)

(Total 40 marks)

Please Turn Over

2. (a) Explain what is meant by the first-order theory of arithmetic for the natural numbers.

(6 marks)

(b) Gödel proved that the first-order theory of arithmetic cannot be finitely axiomatised. Explain what this means, and outline the method by which he proved it.

(10 marks)

(c) Peano Arithmetic is the First-order Theory defined by the axioms:

(S1) $\forall x \neg(sx = 0)$
(S2) $\forall x \forall y(sx = sy \rightarrow x = y)$
(A1) $\forall x(x + 0 = x)$
(A2) $\forall x \forall y(x + sy = s(x + y))$
(M1) $\forall x(x * 0 = 0)$
(M2) $\forall x \forall y(x * sy = (x * y) + x)$

together with the axiom schema

(Ind) $\Phi(0) \wedge \forall x(\Phi(x) \rightarrow \Phi(x + 1)) \rightarrow \forall x(\Phi(x))$

Show that the formula $\forall x(s0 * x = x)$ is a theorem of Peano Arithmetic.

(9 marks)

(d) What can be said about the relation between Peano Arithmetic and the first-order theory of arithmetic in the light of Gödel's proof?

(5 marks)

(Total 30 marks)

3. A Turing Machine has tape alphabet Σ and state set $Q = \{0, 1, 2, \ldots, n\}$, where 0 is the halt state and 1 the start state. The operation of the machine is specified by means of three functions defining the next state, new symbol, and displacement for each state/symbol pair:

$$\begin{aligned} T: &\quad Q \times \Sigma \to Q \\ U: &\quad Q \times \Sigma \to \Sigma \\ D: &\quad Q \times \Sigma \to \{-1, 0, 1\} \end{aligned}$$

(a) The specification above is supplemented by the rule

$$\forall x \in \Sigma(T(0, x) = 0 \wedge U(0, x) = x \wedge D(0, x) = 0).$$

Explain the effect of this rule and why it is needed.

(3 marks)

(b) A configuration of the Turing machine at any computation step can be specified in a natural way as a quadruple $(q, a, b, c) \in Q \times \Sigma \times \Sigma^* \times \Sigma^*$. Explain the meaning of each element of such a quadruple, relating your explanation to the usual informal vocabulary of "tape", "scanned square", etc.

(6 marks)

(c) For a given initial configuration, the configuration after n steps is given by a quadruple $C(n) = (q(n), a(n), b(n), c(n))$ of the type specified in (b). Describe in detail how $C(n + 1)$ is determined from $C(n)$.

(10 marks)

(d) In terms of the function C introduced in (c), write down the conditions that

(i) the machine reaches the halt state after n computation steps;

(ii) the machine eventually reaches the halt state.

(4 marks)

(e) Explain in outline how the above considerations can be used to construct a first-order inference which is valid if and only if the machine halts. Explain the connection between the halting problem for Turing machines and the *Entscheidungsproblem* for first-order logic.

(7 marks)

(Total 30 marks)

Please Turn Over

4. (a) Consider the following problem: Given a finite set of integers S and an integer n, determine whether or not S contains a subset whose members sum to n. Explain what is meant by the complexity class NP, and show that this problem belongs to that class.

(6 marks)

(b) A subclass of NP is the class NP-complete. How is this class defined? Give an example of one problem in NP (other than SAT) which is also in NP-complete, and one problem in NP which is *not* in NP-complete.

(6 marks)

(c) The first problem to be identified as NP-complete was SAT, the Satisfiability Problem for the Propositional Calculus. Explain what this problem is, and outline the method by which it was shown to be NP-complete.

(8 marks)

(d) Explain how the NP-completeness of SAT can be used to establish the existence of other NP-complete problems.

(5 marks)

(e) Discuss the implications of NP-completeness for practical computing concerns.

(5 marks)

(Total 30 marks)

5. *Answer ONE of the following:* Discuss arguments for and against *one* of the following statements:

(a) All computation consists in the transformation of strings over a finite input alphabet into strings over a finite output alphabet in such a way that the relationship between the input strings and the output strings can be expressed by means of a finite set of first-order formulae.

(b) In view of Gödel's incompleteness theorems, the view that human intelligence arises solely as a result of the computational activity of the brain cannot be upheld.

(30 marks)

(Total 30 marks)

End of Paper