COM3412

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING, COMPUTING AND MATHEMATICS

Logic and Computation

TWO HOURS

Answer question 1, and two out of the four other questions.
Question 1 is worth 40 marks. Other questions are worth 30 marks each.

Candidates are advised to spend FORTY-FIVE minutes on question 1
and THIRTY-FIVE minutes on other questions.

No electronic calculators of any sort are to be used during the course of this examination.

Compulsory question

1. (a) Explain what is meant by

- a first-order language;
- an interpretation for a first-order language;
- a model for a set of first-order formulae.

How is the logical notion of *validity* defined in terms of these concepts?

(12 marks)

(b) The inference rules for Natural Deduction (listed in the appendix to this question paper) include *introduction rules* and *elimination rules* for each of the *logical constants*. Explain the meanings of the terms in italics.

(6 marks)

(c) State the semantic rules for disjunction ($\vee$) and material conditional ($\rightarrow$) and use these, together with the definition of validity given in (a), to justify the $\rightarrow$-introduction and $\vee$-elimination rules.

(10 marks)

(d) Use Natural Deduction to establish the validity of the inference

$$\begin{array}{|l} A \rightarrow C \\ B \rightarrow C \\ \hline A \vee B \rightarrow C \end{array}$$

(12 marks)

(Total 40 marks)

2. (a) What is meant by a *proof system* for a logic? Describe two examples of proof systems.

Explain the terms *soundness*, *completeness*, and *decidability* in relation to proof systems.

(12 marks)

(b) A logic is said to be *compact* if, whenever a formula ϕ is logically entailed by a set of formulae Σ, it is also entailed by some finite subset $\Sigma_0 \subseteq \Sigma$.

Show that if there is a sound and complete proof system for a particular logic, then that logic must be compact.

(8 marks)

(c) First-order logic is said to be *finitary*, meaning that it cannot contain formulae of infinite length. Let us consider the effect of dropping this restriction. In *infinitary first-order logic*, we can form infinite formulae, such as the conjunction

$$\phi_0 \wedge \phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \cdots,$$

where $\{\phi_i \mid i \in \mathbb{N}\}$ is an infinite set of distinct formulae. This may be abbreviated to

$$\bigwedge_{i=0}^{\infty} \phi_i.$$

By considering the entailment

$$\{\phi_i \mid i \in \mathbb{N}\} \models \bigwedge_{i=0}^{\infty} \phi_i,$$

show that infinitary first-order logic is not compact.

(10 marks)

(Total 30 marks)

Please Turn Over

3. (a) Explain what is meant by

i. The Halting Problem for Turing Machines,

ii. The Decision Problem for First-order Logic,

and discuss the practical importance of these problems.

(12 marks)

(b) Outline the steps by which it can be shown that these two problems are equivalent.

(12 marks)

(c) What is our current state of knowledge regarding the solvability or otherwise of these problems?

(6 marks)

(Total 30 marks)

4. The first-order theory of a commutative, associative operation f with identity i includes the following axioms:

$$\begin{array}{ll} (1) & \forall x \forall y (f(x,y) = f(y,x)) \\ (2) & \forall x \forall y \forall z (f(x, f(y,z)) = f(f(x,y),z)) \\ (3) & \forall x (f(x,i) = x) \end{array}$$

Three examples of such operations are

(Ex1) Addition on the set of real numbers.
(Ex2) Multiplication on the set of real numbers.
(Ex3) Union on the set of subsets of the real numbers.

(a) For each of the examples (Ex1)–(Ex3), state, with justification, what element is denoted by the constant symbol i appearing in axiom 3.

(4 marks)

(b) Some models of the axioms satisfy the additional formula

$$(4) \quad \forall x (f(x,x) = x).$$

For each of the examples (Ex1–Ex3), determine whether or not (4) holds.

(4 marks)

(c) Show that if all four formulae (1)–(4) hold in some interpretation then the formula

$$\forall x \forall y (f(x, f(x,y)) = f(y, f(y,x)))$$

must also hold.

(6 marks)

(d) Another formula which may hold for some models of the axioms is

$$(5) \quad \forall x \exists y (f(x,y) = i).$$

Again determine, stating your reasons, which of the interpretations (Ex1)–(Ex3) satisfy (5).

(4 marks)

Continued . . .

(e) By considering the expression $f(x, f(x, y))$ in two different ways, where $f(x, y) = i$, show that if (1)–(5) all hold, then the domain contains only one element.

(6 marks)

(f) State what is meant by a *complete axiomatisation* of a first-order theory. Explain why the examples in this question show that the axiom set $\{$1,2,3$\}$ cannot be a complete axiomatisation of any theory. What about the set $\{$1,2,3,4,5$\}$?

(6 marks)

(Total 30 marks)

5. *Essay question*

EITHER

(a) Write an essay on Cook's Theorem and the theory of NP-completeness, including a clear statement of what Cook proved, an accurate outline of the method by which he proved it, a correct definition of the class NP-complete, and a discussion of NP-complete problems and their implications for practical computing.

OR

(b) Write an essay on the Church-Turing Thesis, including a clear statement of what the thesis states, an account of the reasons why it is generally held to be true, the implications of this for practical computing, and a discussion of what would have to happen for the thesis to come to be discredited.

(30 marks)

(Total 30 marks)

Please Turn Over

APPENDIX: Natural Deduction inference rules

Rule	Inputs	Output
rep	ϕ	ϕ
$\wedge$-intro	ϕ, ψ	$\phi \wedge \psi$
$\wedge$-elim-left	$\phi \wedge \psi$	ϕ
$\wedge$-elim-right	$\phi \wedge \psi$	ψ
$\vee$-intro-left	ϕ	$\phi \vee \psi$
$\vee$-intro-right	ϕ	$\psi \vee \phi$
$\vee$-elim	$\phi \vee \psi$, $\{\phi\} \vdash \chi, \{\psi\} \vdash \chi$	χ
$\rightarrow$-intro	$\{\phi\} \vdash \psi$	$\phi \rightarrow \psi$
$\rightarrow$-elim	$\phi, \phi \rightarrow \psi$	ψ
$\leftrightarrow$-intro	$\{\phi\} \vdash \psi, \{\psi\} \vdash \phi$	$\phi \leftrightarrow \psi$
$\leftrightarrow$-elim-left	$\phi, \phi \leftrightarrow \psi$	ψ
$\leftrightarrow$-elim-right	$\psi, \phi \leftrightarrow \psi$	ϕ
$\neg$-intro	$\{\phi\} \vdash \psi, \{\phi\} \vdash \neg\psi$	$\neg\phi$
$\neg$-elim	$\neg\neg\phi$	ϕ

End of Paper