ECM3404: Logic and Computation

Logical Preliminaries II: Natural Deduction

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What is Natural Deduction?

Natural Deduction is a *proof method* that can be used for various systems of formal logic, including FOPC.

It provides a systematic way of deriving valid conclusions from premises expressed in logical notation.

It uses a set of primitive *inference rules* which represent the most basic forms of valid inference.

Inference rules can be applied to the premises to derive new formulae which validly follow from the premises.

By iteration, we can eventually derive all the logical consequences of the premises.

Inference Rules

Inference rules are of two kinds: *simple rules* and *subderivation rules*.

A simple rule allows you to derive some formula directly from some other formulae.

Example: The *if-elimination* rule says that from ϕ and $\phi \rightarrow \psi$ you can derive ψ.

A subderivation rule allows you to derive a formula so long as you can perform certain other derivations.

Example: The *if-introduction* rule says that if you can derive ψ from the premises together with some additional assumption ϕ then you can derive $\phi \rightarrow \psi$ from the premises on their own.

Introduction and Elimination Rules

For each *logical constant* (i.e., connective or quantifier) there is an *introduction rule* and an *elimination rule*.

The introduction rule for a connective * is a rule which tells you what you need to do in order to derive a formula containing *.

The elimination rule for * tells you how to derive, from a premise which contains *, a conclusion which does not contain (that occurrence of) *.

NOTE: The distinction between introduction and elimination rules is independent of the distinction between simple and subderivation rules.

NATURAL DEDUCTION: SIMPLE RULES

Rules for Conjunction

and-introduction
From ϕ and ψ we can derive $\phi \wedge \psi$.

$$\wedge\text{-intro} \quad \begin{array}{|l} \phi \\ \psi \\ \hline \phi \wedge \psi \end{array}$$

and-elimination
From $\phi \wedge \psi$ we can derive
(1) ϕ (and-elimination-left)
(2) ψ (and-elimination-right)

$$\wedge\text{-elim-l} \quad \begin{array}{|l} \phi \wedge \psi \\ \hline \phi \end{array} \qquad \wedge\text{-elim-r} \quad \begin{array}{|l} \phi \wedge \psi \\ \hline \psi \end{array}$$

How to Construct a Derivation

From $A \wedge B$ we shall derive $B \wedge A$.

We do this by extracting A and B from $A \wedge B$ (using the and-elimination rules) and then putting them together in the opposite order (using and-introduction).

1.	$A \wedge B$	(premise)
2.	A	$(1, \wedge$-elim-l$)$
3.	B	$(1, \wedge$-elim-r$)$
4.	$B \wedge A$	$(3, 2, \wedge$-intro$)$

Note: Each line consists of a *line number*, a *formula*, and a *justification*. These are all essential parts of the derivation.

Exercise: Derive $A \wedge (B \wedge C)$ from $(A \wedge B) \wedge C$.

Solution to the Exercise

1.	$(A \wedge B) \wedge C$	(premise)
2.	$A \wedge B$	$(1, \wedge$-elim-l$)$
3.	C	$(1, \wedge$-elim-r$)$
4.	A	$(2, \wedge$-elim-l$)$
5.	B.	$(2, \wedge$-elim-r$)$
6.	$B \wedge C$	$(5, 3, \wedge$-intro$)$
7.	$A \wedge (B \wedge C)$	$(4, 6, \wedge$-intro$)$

Inputs and Outputs

Each inference rule requires a specific form of input and delivers a specific form of output. You can only use a rule if inputs of the right form are available; and the output you derive must be of the right form also.

Example: $\wedge$-intro requires *two* formulae as input, and delivers as output a formula which is the conjunction of the two inputs, in the order that they are specified.

More simple rules

or-introduction
From ϕ we can derive
(1) $\phi \vee \psi$ (or-introduction-right)
(2) $\psi \vee \phi$ (or-introduction-left)

$\vee$-intro-r

ϕ
$\phi \vee \psi$

$\vee$-intro-l

ϕ
$\psi \vee \phi$

if-elimination
From ϕ and $\phi \rightarrow \psi$ we can derive ψ

$\rightarrow$-elim

ϕ
$\phi \rightarrow \psi$
ψ

Example

Validate the inference
It's raining and it's dark.
If it's raining or it's snowing then it's cloudy.
Therefore it's cloudy.

We shall write this in logical notation as

$Raining \wedge Dark$
$(Raining \vee Snowing) \rightarrow Cloudy$
$Cloudy$

1.	$Raining \wedge Dark$	(premise)
2.	$(Raining \vee Snowing) \rightarrow Cloudy$	(premise)
3.	$Raining$	(1, $\wedge$-elim-l)
4.	$Raining \vee Snowing$	(3, $\vee$-intro-r)
5.	$Cloudy$	(4, 2, $\rightarrow$ -elim)

The same example, using schematic letters

$A \wedge D$
$(A \vee B) \rightarrow C$
C

1.	$A \wedge D$	(premise)
2.	$(A \vee B) \rightarrow C$	(premise)
3.	A	(1, $\wedge$-elim-l)
4.	$A \vee B$	(3, $\vee$-intro-r)
5.	C	(4, 2, $\rightarrow$ -elim)

The remaining simple rules

iff-elimination
(1) From $\phi \leftrightarrow \psi$ and ϕ we can derive ψ (iff-elimination-right)
(2) From $\phi \leftrightarrow \psi$ and ψ we can derive ϕ (iff-elimination-left)

$\leftrightarrow$-elim-l

$\phi \leftrightarrow \psi$
ψ
ϕ

$\leftrightarrow$-elim-r

$\phi \leftrightarrow \psi$
ϕ
ψ

not-elimination
From $\neg\neg\phi$ we can derive ϕ

$\neg$-elim

$\neg\neg\phi$
ϕ

repetition
From ϕ we can derive ϕ

rep

ϕ
ϕ

An example using several rules

$(A \vee B) \rightarrow (C \wedge D)$
$A \wedge C$
$B \vee D$

1.	$(A \vee B) \rightarrow (C \wedge D)$	(premise)
2.	$A \wedge C$	(premise)
3.	A	(2, $\wedge$-elim-l)
4.	$A \vee B$	(3, $\vee$-intro-r)
5.	$C \wedge D$	(1, 4, $\rightarrow$ -elim)
6.	D	(5, $\wedge$-elim-r)
7.	$B \vee D$	(6, $\vee$-intro-l)

NATURAL DEDUCTION: SUBDERIVATION RULES

A subderivation rule: if-introduction

if-introduction (Conditional Proof)
If we can derive ψ from assumption ϕ then we can derive $\phi \rightarrow \psi$ without that assumption.

$\rightarrow$ -intro

ϕ
ψ
$\phi \rightarrow \psi$

How to use if-introduction

Suppose want to prove $\phi \to \psi$.

We must show that if we assume ϕ then we can prove ψ.

We do this in a *subderivation*, i.e., a derivation nested within the main derivation.

The subderivation will start with the assumption ϕ and end with the conclusion ψ.

Then we come out of the subderivation (this is called 'discharging the assumption') and conclude $\phi \to \psi$.

IMPORTANT NOTE: A subderivation must always begin with an assumption; and the only place an assumption can occur is as the first line of a subderivation.

Example

$A \to B$
$B \to C$
———
$A \to C$

1. $A \to B$	(premise)
2. $B \to C$	(premise)
3. SUBDERIVATION	
3.1. A	(assumption)
3.2. B	(3.1, 1, $\to$-elim)
3.3. C	(3.2, 2, $\to$-elim)
4. $A \to C$	(3, $\to$-intro)

Note that the justification in 4 cites the whole subderivation 3 as the input to $\to$-intro.

A derivation with no premises

———
$A \to (B \to A)$

1. SUBDERIVATION	
1.1. A	(assumption)
1.2. SUBDERIVATION	
1.2.1. B	(assumption)
1.2.2. A	(1.1, rep)
1.3. $B \to A$	(1.2, $\to$-intro)
2. $A \to (B \to A)$	(1, $\to$-intro)

IMPORTANT NOTE: From inside a subderivation you can 'look outside' to formulae established before entering the subderivation. But once you have left the subderivation, you cannot 'look inside' to formulae established within the subderivation.

Subderivation rule: not-introduction

not-introduction (Proof by Contradiction)
If we derive a contradiction from assumption ϕ then we have proved $\neg\phi$.

$\neg$-intro

| ϕ
| ———
| ψ

| ϕ
| ———
| $\neg\psi$
———
$\neg\phi$

How to use not-introduction

Suppose we want to prove $\neg\phi$.

We must show that if we assume ϕ then we can derive a contradiction.

We do this using two subderivations: one starting with assumption ϕ and ending with some formula ψ, the other starting with the same assumption ϕ and ending with the negated formula $\neg\psi$.

When we have come out of the second subderivation we can conclude $\neg\phi$.

Example

$A \to B$
$\neg B$
———
$\neg A$

1. $A \to B$	(premise)
2. $\neg B$	(premise)
3. SUBDERIVATION	
3.1. A	(assumption)
3.2. B	(1, 3.1, $\to$-elim)
4. SUBDERIVATION	
4.1. A	(assumption)
4.2. $\neg B$	(2, rep)
5. $\neg A$	(3, 4, $\neg$-intro)

From a contradiction, anything follows

A
$\neg A$
———
B

1. SUBDERIVATION	
1.1. $\neg B$	(assumption)
1.2. A	(premise)
2. SUBDERIVATION	
2.1. $\neg B$	(assumption)
2.2. $\neg A$	(premise)
3. $\neg\neg B$	(1, 2, $\neg$-intro)
4. B	(3, $\neg$-elim)

This example shows that it is not always necessary to declare your premisses at the start of the derivation: you can bring them in as and when you need to use them.

Subderivation rule: or-elimination

or-elimination (Reasoning by cases) If we can derive χ from both ϕ and ψ separately, then we can derive χ from $\phi \vee \psi$.

$\vee$-elim

| ϕ
| ———
| χ

| ψ
| ———
| χ

$\phi \vee \psi$
———
χ

How to use or-elimination

Suppose we want to prove χ from $\phi \vee \psi$.

We must show that we can derive χ from ϕ, and also that we can derive χ from ψ.

For this we need two subderivations, one with assumption ϕ and conclusion χ, the other with assumption ψ and conclusion χ.

When we have come out of the second subderivation we can conclude χ from our initial premise $\phi \vee \psi$.

Example

$A \vee B$

$B \vee A$

1. $A \vee B$	(premise)
2. SUBDERIVATION	
2.1. A	(assumption)
2.2. $B \vee A$	(2.1, $\vee$-intro-l)
3. SUBDERIVATION	
3.1. B	(assumption)
3.2. $B \vee A$	(3.1, $\vee$-intro-r)
4. $B \vee A$	(1, 2, 3, $\vee$-elim)

A harder example using all three subderivation rules

$\neg A \vee B$

$A \rightarrow B$

We want to derive conclusion $A \rightarrow B$ — this suggests use of if-introduction.

We want to use premise $\neg A \vee B$ — this suggests use of or-elimination.

We will find ourselves needing to use not-introduction; this is suggested by the 'last grasp' heuristic:

If all else fails, try not-introduction.

1. $\neg A \vee B$	(premise)
2. SUBDERIVATION	
2.1. A	(assumption)
2.2. SUBDERIVATION	
2.2.1. $\neg A$	(assumption)
2.2.2. SUBDERIVATION	
2.2.2.1. $\neg B$	(assumption)
2.2.2.2. A	(2.1, rep)
2.2.3. SUBDERIVATION	
2.2.3.1. $\neg B$	(assumption)
2.2.3.2. $\neg A$	(2.2.1, rep)
2.2.4. $\neg\neg B$	(2.2.2, 2.2.3, $\neg$-intro)
2.2.5. B	(2.2.4, $\neg$-elim)
2.3. SUBDERIVATION	
2.3.1. B	(assumption)
2.3.2. B	(2.3.1, rep)
2.4. B	(1, 2.2, 2.3, $\vee$-elim)
3. $A \rightarrow B$	(2, $\rightarrow$ -intro)

The final subderivation rule: iff-introduction

iff-introduction
If we can derive ϕ and ψ from each other then we can prove $\phi \leftrightarrow \psi$.

$\leftrightarrow$ -intro

ϕ

ψ

ψ

ϕ

$\phi \leftrightarrow \psi$