ECM3404: Logic and Computation

Solutions to the Tutorial exercise on NP-completeness and Problem Reduction

1. First, introduce proposition letters as follows:

   - For $1 \le i < j \le n$, let E_{ij} say that $e_{ij} \in E$ (i.e., vertices v_i and v_j are joined in the graph).
   - For $1 \le i, j \le n$ let C_{ij} say that the ith vertex visited in the circuit is v_j.

   Then for the given instance of HAM we need the following clauses:

   (a) For $1 \le i < j \le n$, include E_{ij} if $e_{ij} \in E$, and $\neg E_{ij}$ otherwise. This gives a total of $O(n^2)$ literals, completely defining the graph.

   (b) For $i = 1, \ldots, n$ include the clause

   $$C_{i1} \vee C_{i2} \vee \cdots \vee C_{in},$$

   saying that the ith vertex of the circuit is one of the vertices of the graph. This gives altogether $O(n^2)$ literals.

   (c) For $i = 1, \ldots, n$, $1 \le j < k \le n$ include the clause

   $$\neg(C_{ij} \wedge C_{ik}),$$

   saying that at most one vertex occurs at the ith position of the circuit. This gives $O(n^3)$ literals.

   (d) For $i = 1, \ldots, n$, include the clause

   $$C_{1i} \vee C_{2i} \vee \cdots \vee C_{ni},$$

   saying that vertex i is visited at least once. This gives $O(n^2)$ literals.

   (e) For $1 \le i < j \le n$, $1 \le k \le n$, include the clause

   $$\neg(C_{ik} \wedge C_{jk})$$

   saying that vertex k is not visited more than once. This gives $O(n^3)$ literals. (See note below.)

   (f) For $1 \le j < k \le n$, include the clauses

   $$C_{nj} \wedge C_{1k} \to E_{jk}, \; C_{nk} \wedge C_{1j} \to E_{jk}.$$

   saying that the last vertex in the circuit is joined to the first by an edge in the graph. This gives $O(n^2)$ literals.

(g) For $i = 1, \ldots, n-1$, $1 \leq j < k \leq n$, include the clauses

$$C_{ij} \wedge C_{(i+1)k} \rightarrow E_{jk},\ C_{ik} \wedge C_{(i+1)j} \rightarrow E_{jk},$$

saying that remaining vertices occurring at consecutive positions in the circuit are joined by an edge in the graph. This gives $O(n^3)$ literals.

The total number of literals is $O(n^3)$.

NOTE: It is not really necessary to include the clauses under (e) since they are implied by the ones already introduced; however, we include them here since they simplify the reasoning in the example below. To see why they are implied by the other clauses, suppose we have $C_{ik} \wedge C_{jk}$ (where $i < j$). By the (b) clauses, this means we have $\neg C_{il}$ and $\neg C_{jl}$ for all $l \neq k$. By the (d) clauses this implies that for each $l \neq k$ there is an $m \notin \{i, j\}$ for which we have C_{ml}. Thus there are $n-2$ available ms to be associated with $n-1$ different ls. This implies that for some m there are distinct l, l' for which we have $C_{ml} \wedge C_{ml'}$, contradicting the (c) clauses. Hence we must have $\neg(C_{ik} \wedge C_{jk})$.

Here are the clauses needed for the given instance. We'll follow the order given above.

(a) (1) E_{12}
(2) $\neg E_{13}$
(3) E_{23}

(b) (4) $C_{11} \vee C_{12} \vee C_{13}$
(5) $C_{21} \vee C_{22} \vee C_{23}$
(6) $C_{31} \vee C_{32} \vee C_{33}$

(c) (7) $\neg(C_{11} \wedge C_{12})$
(8) $\neg(C_{11} \wedge C_{13})$
(9) $\neg(C_{12} \wedge C_{13})$
(10) $\neg(C_{21} \wedge C_{22})$
(11) $\neg(C_{21} \wedge C_{23})$
(12) $\neg(C_{22} \wedge C_{23})$
(13) $\neg(C_{31} \wedge C_{32})$
(14) $\neg(C_{31} \wedge C_{33})$
(15) $\neg(C_{32} \wedge C_{33})$

(d) (16) $C_{11} \vee C_{21} \vee C_{31}$
(17) $C_{12} \vee C_{22} \vee C_{32}$
(18) $C_{13} \vee C_{23} \vee C_{33}$

(e) (19) $\neg(C_{11} \wedge C_{21})$
(20) $\neg(C_{11} \wedge C_{31})$
(21) $\neg(C_{21} \wedge C_{31})$
(22) $\neg(C_{12} \wedge C_{22})$
(23) $\neg(C_{12} \wedge C_{32})$
(24) $\neg(C_{22} \wedge C_{32})$
(25) $\neg(C_{13} \wedge C_{23})$
(26) $\neg(C_{13} \wedge C_{33})$
(27) $\neg(C_{23} \wedge C_{33})$

(f) (28) $C_{31} \wedge C_{12} \rightarrow E_{12}$
(29) $C_{32} \wedge C_{11} \rightarrow E_{12}$
(30) $C_{31} \wedge C_{13} \rightarrow E_{13}$
(31) $C_{33} \wedge C_{11} \rightarrow E_{13}$
(32) $C_{32} \wedge C_{13} \rightarrow E_{23}$
(33) $C_{33} \wedge C_{12} \rightarrow E_{23}$

(g) (34) $C_{11} \wedge C_{22} \rightarrow E_{12}$
(35) $C_{12} \wedge C_{21} \rightarrow E_{12}$
(36) $C_{11} \wedge C_{23} \rightarrow E_{13}$
(37) $C_{13} \wedge C_{21} \rightarrow E_{13}$
(38) $C_{12} \wedge C_{23} \rightarrow E_{23}$
(39) $C_{13} \wedge C_{22} \rightarrow E_{23}$
(40) $C_{21} \wedge C_{32} \rightarrow E_{12}$
(41) $C_{22} \wedge C_{31} \rightarrow E_{12}$
(42) $C_{21} \wedge C_{33} \rightarrow E_{13}$
(43) $C_{23} \wedge C_{31} \rightarrow E_{13}$
(44) $C_{22} \wedge C_{33} \rightarrow E_{23}$
(45) $C_{23} \wedge C_{32} \rightarrow E_{23}$

We now prove that this set of clauses is unsatisfiable.

By (2) and (36) we have $\neg(C_{11} \wedge C_{23})$;
by (2) and (43) we have $\neg(C_{23} \wedge C_{31})$.
Together these imply $C_{23} \rightarrow \neg(C_{11} \vee C_{31})$, which by (16) implies $C_{23} \rightarrow C_{21}$.
This, together with (11), implies $\neg C_{23}$.
Similarly, (2) with (37) and (42) implies $C_{21} \rightarrow \neg(C_{13} \vee C_{33})$, which with (18) implies $C_{21} \rightarrow C_{23}$.
This, with (11), implies $\neg C_{21}$.
We now have $\neg C_{21} \wedge \neg C_{23}$, which by (5) implies C_{22}.
By (22), this implies $\neg C_{12}$, so by (4) we have either C_{11} or C_{13}.
Suppose we have C_{11}. By (20) this means we have $\neg C_{31}$. Since we have C_{22}, by (24) we have $\neg C_{32}$. Hence by (6) we have C_{33}, giving us $C_{11} \wedge C_{33}$. This is ruled out by (2) and (31). Hence we do not have C_{11}.
We must therefore have C_{13}. By (26) this means we have $\neg C_{33}$. Since we already have $\neg C_{32}$, we must have C_{31} by (6). giving us $C_{13} \wedge C_{31}$. But this is ruled out by (2) and (30).

Hence the clauses are unsatisfiable.