ECM3404: Logic and Computation

Solutions to the Group Meeting exercise on Peano Arithmetic

1. $\forall x(x * ss0 = x + x)$

   - $$\begin{aligned} x * ss0 &= (x * s0) + x && \text{(M2, } y/s0\text{)} \\ &= (x * 0 + x) + x && \text{(M2, } y/0\text{)} \\ &= (0 + x) + x && \text{(M1)} \\ &= x + x && \text{(Result proved on question sheet, } 0 + x = x\text{)} \end{aligned}$$

2. $\forall x \forall y(sx + y = s(x + y))$

   - **Base Case** ($y = 0$)
     $$\begin{aligned} sx + 0 &= sx && \text{(A1, } x/sx\text{)} \\ &= s(x + 0) && \text{(A1)} \end{aligned}$$
   - **Induction Step** (from $y = a$ to $y = sa$)
     From the Induction Hypothesis $sx + a = s(x + a)$ we must derive $sx + sa = s(x + sa)$.
     $$\begin{aligned} sx + sa &= s(sx + a) && \text{(A2, } x/sx, y/a\text{)} \\ &= s(s(x + a)) && \text{(from induction hyposthesis)} \\ &= s(x + sa) && \text{(A2, } y/a\text{)} \end{aligned}$$

3. $\forall x \forall y(x + y = y + x)$

   - **Base Case** ($y = 0$)
     $$\begin{aligned} x + 0 &= x && \text{(A1)} \\ &= 0 + x && \text{(Result proved on question sheet)} \end{aligned}$$
   - **Induction Step** (from $y = a$ to $y = sa$)
     From the Induction Hypothesis $x + a = a + x$ we must derive $x + sa = sa + x$.
     $$\begin{aligned} x + sa &= s(x + a) && \text{(A2, } y/a\text{)} \\ &= s(a + x) && \text{(from induction hypothesis)} \\ &= sa + x && \text{(from result of part (b), } x/a, y/x\text{)} \end{aligned}$$