ECM3404: Logic and Computation

Tutorial, 24th February 2009

Peano Arithmetic

Peano arithmetic (PA) is the first-order theory defined by the following axioms:

S1 $\forall x \neg (sx = 0)$

S2 $\forall x \forall y (sx = sy \rightarrow x = y)$

A1 $\forall x (x + 0 = x)$

A2 $\forall x \forall y (x + sy = s(x + y))$

M1 $\forall x (x * 0 = 0)$

M2 $\forall x \forall y (x * sy = (x * y) + x)$

Ind $\Phi(0) \wedge \forall x (\Phi(x) \rightarrow \Phi(sx)) \rightarrow \forall x (\Phi(x))$

A theorem of PA is

$$\forall x (0 + x = x).$$

We prove it using the following instance of the induction schema **Ind**:

(1) $0 + 0 = 0 \wedge \forall x (0 + x = x \rightarrow 0 + sx = sx) \rightarrow \forall x (0 + x = x)$.

Note that the consequent here is the theorem to be proved, so it suffices to prove the antecedent. This has two conjuncts, called the base case ($0 + 0 = 0$) and the induction step ($\forall x (0 + x = x \rightarrow 0 + sx = sx)$).

The base case follows directly from **A1** with x instantiated to 0:

(2) $0 + 0 = 0$.

For the induction step, we assume (using an arbitrary constant a)

(3) $0 + a = a$.

By $\forall$-elim we can instantiate **A2** with $x = 0, y = a$ to give

(4) $0 + sa = s(0 + a)$

From (3), we can substitute a for $0 + a$ in (4) to give

(5) $0 + sa = sa$.

Since we've derived (5) from (3) we have proved (by $\rightarrow$-intro)

(6) $0 + a = a \rightarrow 0 + sa = sa$.

Since a was an arbitrary constant we can write this as

(7) $\forall x(0 + x = x \rightarrow 0 + sx = sx)$

By $\wedge$-intro and $\rightarrow$-elim, (1), (2), and (7) imply

(8) $\forall x(0 + x = x)$,

as required.

All this may be set out more compactly as follows:

- **Base Case** ($x = 0$)

  $0 + 0 \;\; = \;\; 0$ (A1, $x/0$)

- **Induction Step** (from $x = a$ to $x = sa$)
  From the Induction hypothesis $0 + a = a$ we must prove $0 + sa = sa$.

  $$\begin{aligned} 0 + sa &= s(0 + a) && \text{(A2, } x/0, y/a\text{)} \\ &= sa && \text{(from the induction hypothesis)} \end{aligned}$$

Exercise. Prove the following theorems of PA:

1. $\forall x(x * ss0 = x + x)$
2. $\forall x \forall y(sx + y = s(x + y))$
3. $\forall x \forall y(x + y = y + x)$

Hint: You don't need to use **Ind** for number (1), but you will need to use the theorem we've just proved.