COM3412: Logic and Computation

Exercise on Peano Arithmetic

2nd March 2009

Use Peano's axioms to derive the following theorems. You may make use of the result of any of the theorems in deriving any of the ones occurring later in the list. Questions 3–5 are from the CA; they do not form part of this exercise, but the results may be used in later questions.

1. Addition is associative:
$$\forall x \forall y \forall z (x + (y + z) = (x + y) + z).$$

2. Cancellation by subtraction:
$$\forall x \forall y \forall z (x + z = y + z \rightarrow x = y).$$

3. Pre-multiplication by zero:
$$\forall x (0 * x = 0).$$

4. Pre-multiplication by a successor:
$$\forall x \forall y (sx * y = (x * y) + y).$$

5. Multiplication is commutative:
$$\forall x \forall y (x * y = y * x).$$

6. Multiplication is distributive over addition:
$$\forall x \forall y \forall z (x * (y + z) = (x * y) + (x * z)).$$

7. Multiplication is associative:
$$\forall x \forall y \forall z (x * (y * z) = (x * y) * z).$$

8. Divisors of zero:
$$\forall x \forall y (x * y = 0 \rightarrow x = 0 \vee y = 0).$$

9. Cancellation by division (hard!):
$$\forall x \forall y \forall z (x * sz = y * sz \rightarrow x = y).$$