ECM3404: Logic and Computation

Answers to Tutorial Questions

10th February 2009

1 Natural Deduction (Propositional Calculus, simple rules)

1.
1.	$A \wedge B$	(premise)
2.	$C \wedge D$	(premise)
3.	A	$(1, \wedge$-elim-l)
4.	D	$(2, \wedge$-elim-r)
5.	$A \wedge D$	$(3, 4, \wedge$-intro)

2.
1.	A	(premise)
2.	$A \vee B$	$(1, \vee$-intro-r)
3.	$A \vee B \rightarrow C$	(premise)
4.	C	$(2, 3, \rightarrow$ -elim)
5.	$D \vee C$	$(4, \vee$-intro-l)

3.
1.	$A \wedge C$	(premise)
2.	A	$(1, \wedge$-elim-l)
3.	$A \rightarrow B$	(premise)
4.	B	$(2, 3, \rightarrow$ -elim)
5.	C	$(1, \wedge$-elim-r)
6.	$B \wedge C$	$(4, 5, \wedge$-intro)
7.	$B \wedge C \rightarrow D$	(premise)
8.	D	$(6, 7, \rightarrow$ -elim)

4.
1.	$A \wedge D$	(premise)
2.	A	$(1, \wedge$-elim-l)
3.	$A \leftrightarrow C$	(premise)
4.	C	$(2, 3, \leftrightarrow$ -elim-r)
5.	D	$(1, \wedge$-elim-r)
6.	$B \leftrightarrow D$	(premise)
7.	B	$(5, 6, \leftrightarrow$ -elim-l)
8.	$B \wedge C$	$(7, 4, \wedge$-intro)

5.

1.	$A \wedge B$	(premise)
2.	B	$(1, \wedge\text{-elim-r})$
3.	$B \to C \wedge E$	(premise)
4.	$C \wedge E$	$(2, 3, \to \text{-elim})$
5.	A	$(1, \wedge\text{-elim-l})$
6.	C	$(4, \wedge\text{-elim-l})$
7.	$A \wedge C$	$(5, 6, \wedge\text{-intro})$
8.	$A \wedge C \to D$	(premise)
9.	D	$(7, 8, \to \text{-elim})$
10.	E	$(4, \wedge\text{-elim-r})$
11.	$D \wedge E$	$(9, 10, \wedge\text{-intro})$

Note: There are two different styles we can adopt for constructing these proofs. One is to write down all the premises first, and then perform the deductions from them; the other is not to write down a premise until you actually need to use it. The proofs given here are all in the second style. As an illustration of the first style, here is another answer to number 4; it contains exactly the same steps, but in a different order.

1.	$A \wedge D$	(premise)
2.	$A \leftrightarrow C$	(premise)
3.	$B \leftrightarrow D$	(premise)
4.	A	$(1, \wedge\text{-elim-l})$
5.	C	$(2, 4, \leftrightarrow \text{-elim-r})$
6.	D	$(1, \wedge\text{-elim-r})$
7.	B	$(3, 6, \leftrightarrow \text{-elim-l})$
8.	$B \wedge C$	$(7, 5, \wedge\text{-intro})$

2 Propositional Calculus: All Rules

6. 1. SUBDERIVATION

1.1. $A \wedge B$	(assumption)
1.2. A	(1.1, $\wedge$-elim-l)
1.3. $A \to C$	(premise)
1.4. C	(1.2, 1.3, $\to$ -elim)
1.5. B	(1.1, $\wedge$-elim-r)
1.6. $B \to D$	(premise)
1.7. D	(1.5, 1.6, $\to$ -elim)
1.8. $C \wedge D$	(1.4, 1.7, $\wedge$-intro)
2. $A \wedge B \to C \wedge D$	(1, $\to$ -intro)

Commentary: The conclusion $A \wedge B \to C \wedge D$ is a conditional, so we should use $\to$-intro, assuming $A \wedge B$ and deriving $C \wedge D$.

7. 1. SUBDERIVATION

1.1. A	(assumption)
1.2. $A \to B$	(premise)
1.3. B	(1.1, 1.2, $\to$ -elim)
1.4. $A \wedge B$	(1.1, 1.3, $\wedge$-intro)
1.5. $A \wedge B \to C$	(premise)
1.6. C	(1.4, 1.5, $\to$ -elim)
2. $A \to C$	(1, $\to$ -intro)

Commentary: The conclusion $A \to C$ is a conditional, so we should use $\to$-intro, assuming A and deriving C.

8. 1. SUBDERIVATION

1.1. A	(assumption)
1.2. $A \to (B \to C)$	(premise)
1.3. $B \to C$	(1.1, 1.2, $\to$ -elim)
1.4. B	(premise)
1.5. C	(1.4, 1.3, $\to$ -intro)
2. $A \to C$	(1, $\to$ -intro)

9. 1. SUBDERIVATION

1.1. B	(assumption)
1.2. SUBDERIVATION	
1.2.1. A	(assumption)
1.2.2. $A \to (B \to C)$	(premise)
1.2.3. $B \to C$	(1.2.1, 1.2.2, $\to$ -elim)
1.2.4. C	(1.1, 1.2.3, $\to$ -elim)
1.3. $A \to C$	(1.2, $\to$ -intro)
2. $B \to (A \to C)$	(1, $\to$ -intro)

Commentary: The conclusion $B \to (A \to C)$ is a conditional, we should use $\to$-intro, assuming B and deriving $A \to C$. To derive $A \to C$ we also need $\to$-intro, assuming A and deriving C (as in the previous question). This requires a subderivation nested inside the one we've already started. Note that from within the subderivation 1.2 we can refer to formulae that are in force just before we enter the subderivation — in particular to B at 1.1 which we invoke in the justification for 1.2.4. What we *cannot* do is to refer to formulae inside a subderivation from outside, since the assumption governing the subderivation does not apply outside it.

10. 1. SUBDERIVATION

1.1. A	(assumption)
1.2. SUBDERIVATION	
1.2.1. B	(assumption)
1.2.2. $A \to C$	(premise)
1.2.3. C	(1.1, 1.2.2, $\to$ -elim)
1.3. $B \to C$	(1.2, $\to$ -intro)
2. $A \to (B \to C)$	(1, $\to$ -intro)

Commentary: Note that the assumption B doesn't do any work in the derivation of C from A. This doesn't matter: the inference is still valid. If C follows from A then it must follow from A together with B, which means that $B \to C$ follows from A.

11.

1. $A \vee B \to C$	(premise)
2. SUBDERIVATION	
2.1. A	(assumption)
2.2. $A \vee B$	(2.1, $\vee$-intro-r)
2.3. C	(1, 2.2, $\to$ -elim)
3. $A \to C$	(2, $\to$ -intro)
4. SUBDERIVATION	
4.1. B	(assumption)
4.2. $A \vee B$	(4.1, $\vee$-intro-l)
4.3. C	(1, 4.2, $\to$ -elim)
5. $B \to C$	(4, $\to$ -intro)
6. $(A \to C) \wedge (B \to C)$	(3, 4, $\wedge$-intro)

Commentary: The conclusion is a conjunction, so we should derive this by deriving the two conjuncts and joining them using $\wedge$-intro. Both conjuncts are conditionals, so we need to use $\rightarrow$-intro to derive them; this requires two subderivations, one for each conjunct.

12. 1. SUBDERIVATION
    1.1. A (assumption)
    1.2. $A \rightarrow B$ (premise)
    1.3. B (1.1, 1.2, $\rightarrow$ -elim)
    2. SUBDERIVATION
    2.1. B (assumption)
    2.2. $B \rightarrow A$ (premise)
    2.3. A (2.1, 2.2, $\rightarrow$ -elim)
    3. $A \leftrightarrow B$ (1, 2, $\leftrightarrow$ -intro)

    Commentary: The conclusion $A \leftrightarrow B$ is a biconditional, so we should use $\leftrightarrow$-intro. This requires two subderivations, one deriving B from A, and one deriving A from B.

13. 1. SUBDERIVATION
    1.1. A (assumption)
    1.2. $A \rightarrow \neg A$ (premise)
    1.3. $\neg A$ (1.1, 1.2, $\rightarrow$ -elim)
    2. SUBDERIVATION
    2.1. A (assumption)
    2.2. A (2.1, rep)
    3. $\neg A$ (1, 2, $\neg$-intro)

    Commentary: The conclusion $\neg A$ is a negation, suggesting the use of $\neg$-intro. This requires us to derive a contradiction from A; we do this using two subderivations, both with assumption A, but with contradictory conclusions. Inspection of the premise suggests that A and $\neg A$ would be a suitable contradiction to go for.

14. 1. SUBDERIVATION
    1.1. A (assumption)
    1.2. $A \rightarrow B \wedge C$ (premise)
    1.3. $B \wedge C$ (1.1, 1.2, $\rightarrow$ -elim)
    1.4. B (1.3, $\wedge$-elim-l)
    1.5. $B \rightarrow D$ (premise)
    1.6. D (1.4, 1.5, $\rightarrow$ -elim)
    1.7. $B \wedge D$ (1.4, 1.6, $\wedge$-intro)
    2. SUBDERIVATION
    2.1. A (assumption)
    2.2. $\neg(B \wedge D)$ (premise)
    3. $\neg A$ (1, 2, $\neg$-intro)

    Commentary: The conclusion $\neg A$ is a negation, so we should use $\neg$-introduction, assuming A and deriving a contradiction. What contradiction? The only negation present in the premises is $\neg(B \wedge D)$; if we can derive $B \wedge D$ from A then we will have a contradiction. Inspection of the other two premises shows that we can do this.

15. 1. SUBDERIVATION
    1.1. A (assumption)
    1.2. $(A \to B) \vee C$ (premise)
    1.3. SUBDERIVATION
        1.3.1. $A \to B$ (assumption)
        1.3.2. B (1.1, 1.3.1, $\to$ -elim)
        1.3.3. $B \vee C$ (1.3.2, $\vee$-intro-r)
    1.4. SUBDERIVATION
        1.4.1. C (assumption)
        1.4.2. $B \vee C$ (1.4.1, $\vee$-intro-l)
    1.5. $B \vee C$ (1.2, 1.3, 1.4, $\vee$-elim)
2. $A \to B \vee C$ (1, $\to$ -intro)

Commentary: The conclusion $A \to (B \vee C)$ is a conditional, so we should use $\to$-intro, assuming A and deriving $B \vee C$. We note that the premise $(A \to B) \vee C$ is a disjunction, so we should use $\vee$-elim. This tells us that to derive $B \vee C$ from $(A \to B) \vee C$, we must derive it from $A \to B$ and C separately, in two subderivations. If we can do that, then we can assert that it has been derived from the disjunction. All this happens within the subderivation beginning with assumption A.

An alternative route would be to start by setting up the $\vee$-elim and then performing the $\to$-elim twice, once in each subderivation. This works, but is a little less elegant:

1. $(A \to B) \vee C$ (premise)
2. SUBDERIVATION
    2.1. $A \to B$ (assumption)
    2.2. SUBDERIVATION
        2.2.1. A (assumption)
        2.2.2. B (2.1, 2.2.1, $\to$ -elim)
        2.2.3. $B \vee C$ (2.2.2, $\vee$-intro-r)
    2.3. $A \to B \vee C$ (2.2, $\to$ -intro)
3. SUBDERIVATION
    3.1. C (assumption)
    3.2. SUBDERIVATION
        3.2.1. A (assumption)
        3.2.2. C (3.1, rep)
        3.2.3. $B \vee C$ (3.2.2, $\vee$-intro-l)
    3.3. $A \to B \vee C$ (3.2, $\to$ -intro)
4. $A \to B \vee C$ (1, 2, 3, $\vee$-elim)

16. 1. SUBDERIVATION
    1.1. A (assumption)
    1.2. $A \to B \vee C$ (premise)
    1.3. $B \vee C$ (1.1, 1.2, $\to$ -elim)
    1.4. SUBDERIVATION
        1.4.1. B (assumption)
        1.4.2. $B \to D$ (premise)
        1.4.3. D (1.4.1, 1.4.2, $\to$ -elim)
        1.4.4. $C \vee D$ (1.4.3, $\vee$-intro-l)
    1.5. SUBDERIVATION
        1.5.1. C (assumption)
        1.5.2. $C \vee D$ (1.5.1, $\vee$-intro-r)

1.6. $C \vee D$ (1.3, 1.4, 1.5, $\vee$-elim)
2. $A \to C \vee D$ (1, $\to$ -intro)

Commentary: We need to use $\to$-intro to generate $A \to C \vee D$. Note that in the course of the derivation we have to use the $B \vee C$ extracted from the first premise: this requires us to use $\vee$-elim.

17. 1. SUBDERIVATION
    - 1.1. $A \wedge B$ (assumption)
    - 1.2. $A \wedge B \to C \vee D$ (premise)
    - 1.3. $C \vee D$ (1.1, 1.2, $\to$ -elim)
    - 1.4. SUBDERIVATION
        - 1.4.1. C (assumption)
        - 1.4.2. B (1.1, $\wedge$-elim-r)
        - 1.4.3. $B \wedge C$ (1.4.2, 1.4.1, $\wedge$-intro)
        - 1.4.4. $B \wedge C \to E$ (premise)
        - 1.4.5. E (1.4.3, 1.4.4, $\to$ -elim)
        - 1.4.6. $D \vee E$ (1.4.5, $\vee$-intro-l)
    - 1.5. SUBDERIVATION
        - 1.5.1. D (assumption)
        - 1.5.2. $D \vee E$ (1.5.1, $\vee$-intro-r)
    - 1.6. $D \vee E$ (1.3, 1.4, 1.5, $\vee$-elim)

    2. $A \wedge B \to D \vee E$ (1, $\to$ -intro)

    Commentary: Again we are deriving a conditional, so the overall structure of the derivation is a $\to$-intro. But again, we have $C \vee D$ as an intermediate stage, and to use this we need $\vee$-elim.

18. 1. SUBDERIVATION
    - 1.1. $A \wedge B$ (assumption)
    - 1.2. A (1.1, $\wedge$-elim-l)
    - 1.3. $A \to \neg B \vee C$ (premise)
    - 1.4. $\neg B \vee C$ (1.2, 1.3, $\to$ -elim)
    - 1.5. SUBDERIVATION
        - 1.5.1. $\neg B$ (assumption)
        - 1.5.2. SUBDERIVATION
            - 1.5.2.1. $\neg C$ (assumption)
            - 1.5.2.2. B (1.1, $\wedge$-elim-r)
        - 1.5.3. SUBDERIVATION
            - 1.5.3.1. $\neg C$ (assumption)
            - 1.5.3.2. $\neg B$ (1.5.1, rep)
        - 1.5.4. $\neg\neg C$ (1.5.2, 1.5.3, $\neg$-intro)
        - 1.5.5. C (1.5.4, $\neg$-elim)
    - 1.6. SUBDERIVATION
        - 1.6.1. C (assumption)
        - 1.6.2. C (1.6.1, rep)
    - 1.7. C (1.4, 1.5, 1.6, $\vee$-elim)

    2. $A \wedge B \to C$ (1, $\to$ -intro)

    Commentary: We need to derive C from $A \wedge B$ and then use $\to$-intro. We can quickly get to $\neg B \vee C$, which then requires us to use $\vee$-elim: we must derive C both from $\neg B$ and C. The latter is trivial, the former less straightforward. There is no obvious thing to do, so we resort to $\neg$-intro, deriving a contradiction from $\neg C$.