ECM3404: Logic and Computation

Solutions to the Continuous Assessment Exercise, March 2009

1. Key:

Umbrella	You have an umbrella
Coat	You have a coat
Rain	It rains
Snow	It snows
Wet	You get wet
Freeze	You freeze
Cold	You catch a cold

$\neg\text{Umbrella} \rightarrow (\text{Rain} \rightarrow \text{Wet})$
$(\text{Snow} \wedge \neg\text{Coat}) \rightarrow \text{Freeze}$
$(\text{Freeze} \vee \text{Wet}) \rightarrow \text{Cold}$
$((\text{Rain} \vee \text{Snow}) \wedge \neg\text{Cold}) \rightarrow (\text{Umbrella} \vee \text{Coat})$

1. $\neg\text{Umbrella} \rightarrow (\text{Rain} \rightarrow \text{Wet})$ (Premise)
2. $(\text{Snow} \wedge \neg\text{Coat}) \rightarrow \text{Freeze}$ (Premise)
3. $(\text{Freeze} \vee \text{Wet}) \rightarrow \text{Cold}$ (Premise)
4. SUBDERIVATION
   - 4.1. $(\text{Rain} \vee \text{Snow}) \wedge \neg\text{Cold}$ (Assumption)
   - 4.2. $\text{Rain} \vee \text{Snow}$ (4.1, $\wedge$-elim-l)
   - 4.3. SUBDERIVATION
     - 4.3.1. Rain (Assumption)
     - 4.3.2. SUBDERIVATION
       - 4.3.2.1. $\neg$Umbrella (Assumption)
       - 4.3.2.3. Rain $\rightarrow$ Wet (1, 4.3.2.1, $\rightarrow$ -elim)
       - 4.3.2.4. Wet (4.3.1, 4.3.2.3, $\rightarrow$ -elim)
       - 4.3.2.5. Freeze $\vee$ Wet (4.3.2.4, $\vee$-intro-l)
       - 4.3.2.6. Cold (3, 4.3.2.5, $\rightarrow$ -elim)
     - 4.3.3. SUBDERIVATION
       - 4.3.3.1. $\neg$Umbrella (Assumption)
       - 4.3.3.2. $\neg$Cold (4.1, $\wedge$-elim-r)
     - 4.3.4. $\neg\neg$Umbrella (4.3.2, 4.3.3, $\neg$-intro)
     - 4.3.5. Umbrella (4.3.4, $\neg$-elim)
     - 4.3.6. Umbrella $\vee$ Coat (4.3.5, $\vee$-intro-r)
   - 4.4. SUBDERIVATION
     - 4.4.1. Snow (Assumption)
     - 4.4.2. SUBDERIVATION
       - 4.4.2.1. $\neg$Coat (Assumption)
       - 4.4.2.2. Snow $\wedge \neg$Coat (4.4.1, 4.4.2.1, $\wedge$-intro)
       - 4.4.2.3. Freeze (2, 4.4.2.2, $\rightarrow$ -elim)
       - 4.4.2.4. Freeze $\vee$ Wet (4.4.2.3, $\vee$-intro-r)
       - 4.4.2.5. Cold (3, 4.4.2.4, $\rightarrow$ -elim)
     - 4.4.3. SUBDERIVATION
       - 4.4.3.1. $\neg$Coat (Assumption)
       - 4.4.3.2. $\neg$Cold (4.1, $\wedge$-elim-r)
     - 4.4.4. $\neg\neg$Coat (4.4.2, 4.4.3, $\neg$-intro)
     - 4.4.5. Coat (4.4.4, $\neg$-elim)
     - 4.4.6. Umbrella $\vee$ Coat (4.4.5, $\vee$-intro-l)
   - 4.5. Umbrella $\vee$ Coat (4.2, 4.3, 4.4, $\vee$-elim)
5. $((\text{Rain} \vee \text{Snow}) \wedge \neg\text{Cold}) \rightarrow (\text{Umbrella} \vee \text{Coat})$ (4, $\rightarrow$ -intro)

2. (a) Base case ($x = 0$):

$$0 * 0 \quad = \quad 0 \quad \text{(M1}[x/0]\text{)}$$

Induction step (from $x = a$ to $x = sa$):
Assume (IH) $0 * a = 0$.

$$\begin{array}{rcll} 0 * sa & = & 0 * a + 0 & \text{(M2[x/0,y/a])} \\ & = & 0 * a & \text{(A1}[x/0 * a]\text{)} \\ & = & 0 & \text{(IH)} \end{array}$$

(b) Base case ($y = 0$):

$$\begin{array}{rcll} sx * 0 & = & 0 & \text{(M1}[x/sx]\text{)} \\ & = & 0 + 0 & \text{(A1}[x/0]\text{)} \\ & = & x * 0 + 0 & \text{(M1)} \end{array}$$

Induction step (from $y = a$ to $y = sa$):
Assume (IH) $sx * a = x * a + a$.

$$\begin{array}{rcll} sx * sa & = & sx * a + sx & \text{(M2[x/sx,y/a]} \\ & = & (x * a + a) + sx & \text{(IH)} \\ & = & x * a + (a + sx) & \text{(Associativity of } +\text{)} \\ & = & x * a + (sx + a) & \text{(Tutorial q.3, } [x/a, y/sx]\text{)} \\ & = & x * a + s(x + a) & \text{(Tutorial q.2, } [x/x, y/a]\text{)} \\ & = & x * a + (x + sa) & \text{(A2[x/x,y/a])} \\ & = & (x * a + x) + sa & \text{(Associativity of } +\text{)} \\ & = & x * sa + sa & \text{(M2}[x/x, y/a]\text{)} \end{array}$$

(c) Base case ($y = 0$):

$$\begin{array}{rcll} x * 0 & = & 0 & \text{(M1)} \\ & = & 0 * x & \text{(Part (a))} \end{array}$$

Induction step (from $y = a$ to $y = sa$):
Assume (IH) $x * a = a * x$:

$$\begin{array}{rcll} x * sa & = & x * a + x & \text{(M2}[x/x, y/a]\text{)} \\ & = & a * x + x & \text{(IH)} \\ & = & sa * x & \text{(Part (b), } [x/a, y/x]\text{)} \end{array}$$