Basics of complex numbers

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This is a summary of the basic facts about complex numbers. I shall assume that everyone coming to this course **already knows** this material; I provide this only as a reminder and reference.

The set $\mathbf{C}$ of *complex numbers* is defined as $\mathbf{C} = \{x + yi : x, y \in \mathbf{R}\}$ where $i^2 = -1$. Addition, subtraction and multiplication of complex numbers (using $i^2 = -1$) is straightforward.

We represent complex numbers as points in the *Argand diagram* of "complex plane". The complex number $z = x + yi$ is identified with the point whose Cartesian coordinates are (x, y).

The *real part* of $z = x + yi$ is $\operatorname{Re} z = x$, its *imaginary part* is $\operatorname{Im} z = y$ and its *complex conjugate* is $\overline{z} = x - yi$. Then $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{z - w} = \overline{z} - \overline{w}$, $\overline{zw} = \overline{z}\,\overline{w}$ and $\overline{\overline{z}} = z$. Also $z\overline{z} = x^2 + y^2 \ge 0$ and $z\overline{z} = 0$ if and only if $z = 0$. The *absolute value* of z is $|z| = \sqrt{z\overline{z}}$. If $z \ne 0$ and $w = \overline{z}|z|^{-2}$ then $zw = 1$ so that z has a reciprocal (and $\mathbf{C}$ is a field). Note that $|z - w|$ is the distance between points z and w in the Argand diagram.

One basic theorem in complex numbers is the *triangle inequality*: $|z+w| \le |z| + |w|$.

If z is a nonzero complex number then $w = z/|z|$ satisfies $|w| = 1$. So w lies on the *unit circle* in the Argand diagram, that is the circle with centre 0 and radius 1. It follows that there is some real number θ with $w = \cos\theta + i\sin\theta$. We write $e^{i\theta}$ for $\cos\theta + i\sin\theta$ and note that the addition identities for sine and cosine imply that $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$. We can then write $z = re^{i\theta}$ where $r = |z| > 0$ and θ in $\mathbf{R}$. Such a number θ is called an *argument* of z. The argument of z is not unique since $e^{i\theta} = e^{i(\theta+2\pi)}$. However, z has a unique argument θ in the interval $(-\pi, \pi]$ which we call the *principal argument* and denote by $\operatorname{Arg} z$. The general argument of z is $\operatorname{Arg} z + 2k\pi$ where $k \in \mathbf{Z}$.

We define the complex exponential by $\exp(x + iy) = e^x e^{iy} = e^x(\cos y + i\sin y)$ for x. $y \in \mathbf{R}$. Then $\exp(z + w) = \exp(z)\exp(w)$. For non-zero z, the equation $e^w = z$ has the general solution $w = \log|z| + i\arg z + 2k\pi i$ (where

$k \in \mathbf{Z}$). Then $\log|z| + i \operatorname{Arg} z$ is defined to be the *principal logarithm* $\operatorname{Log} z$ of z.

Convergence of sequences and series of complex numbers are defined in much the same way as those of real numbers. A sequence (z_n) of complex numbers converges to a limit z if for all $\varepsilon > 0$ there is N such that $n \geq N$ implies $|z_n - w| < \varepsilon$. Then $\lim_{n\to\infty} z_n = w$ if and only if both $\lim_{n\to\infty} \operatorname{Re} z_n = \operatorname{Re} w$ and $\lim_{n\to\infty} \operatorname{Im} z_n = \operatorname{Im} w$. Sums, differences, products and quotients (under the usual caveats) of convergent complex sequences are convergent. Again, a series $\sum_{n=1}^{\infty} z_n$ converges to z if and only if $\lim_{N\to\infty} \sum_{n=1}^{N} z_n = w$. As with real series, absolute convergence implies convergence.

We also consider complex functions: maps $f : A \to \mathbf{C}$ where $A \subseteq \mathbf{C}$ Limits and continuity for complex functions are defined in the same way as for real functions. For instance $f : A \to \mathbf{C}$ is continuous at $a \in A$ if and only if $\lim_{n\to\infty} f(a_n) = f(a)$ for all sequences (a_n) of points in A with $a_n \to a$. Again, continuity satisfies the same basic properties as for real functions: for example, sums, differences, products, quotients and composites of continuous functions (subject to the usual caveats) are continuous. As a consequence, polynomial functions are continuous, and so are rational functions where they are defined (where the denominator is nonzero).

The complex exponential function exp is continuous on $\mathbf{C}$. Indeed $\exp(z) = \sum_{n=0}^{\infty} z^n/n!$. For real x, as $e^{ix} = \cos x + i \sin x$ and $e^{-ix} = \cos x - i \sin x$ then $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ and $i \sin x = \frac{1}{2}(e^{ix} - e^{-ix})$. We define the complex sine and cosine function using these formulae:

$$\cos z = \frac{\exp(iz) + \exp(-iz)}{2}, \qquad \sin z = \frac{\exp(iz) - \exp(-iz)}{2i}.$$

Then $\cos iz = \frac{1}{2}(\exp(z) + \exp(-z)) = \cosh z$ and $\sin iz = -i\frac{1}{2}(\exp(-z) - \exp(z)) = i \sinh z$. This shows that although the sine and cosine are bounded on $\mathbf{R}$ they are not bounded functions on $\mathbf{C}$.