MAS3006

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING, COMPUTER SCIENCE & MATHEMATICS

DEPARTMENT OF MATHEMATICAL SCIENCES

COMPLEX ANALYSIS

31 May 2005

2:15 p.m. – 4:15 p.m.
Duration: 2 hours

Examiner: Dr M. Saïdi

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline.

Calculators labelled as approved by the Department of Mathematical Sciences may be used.

SECTION A

1. (a) Is the set $H := \{z : \operatorname{Im} z > 0\}$ open, closed, neither or both? Justify your answer. (10)

(b) Determine the subset of the complex plane on which the function $f(z) = \dfrac{e^z}{(z^2+2)}$ is analytic. (5)

(c) Let $z = x + iy$ and $f(z) = (2x + y^2) + i(x^2 - y^2)$. Determine the points $z = x + iy$ for which the derivative $f'(z)$ exists. (5)

(d) Let γ be the circle of centre 0 and radius 1, traversed once anticlockwise. Prove that

$$\left| \int_\gamma \frac{\sin z \, dz}{z^2} \right| \le 2\pi e \, .$$

(5)

(e) Evaluate the integral $\displaystyle\int_\gamma \frac{(z+2)}{(z-1)} dz$ where γ is the circle with centre 0 and radius 2, traversed once anticlockwise. (5)

(f) Find the radius of convergence of the following series $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{e^n}$. (5)

(g) Compute the Taylor series of the function $f(z) = ze^{z^2}$ around $z = 0$ and indicate the radius of convergence of this series. (5)

(h) Compute the integral $\displaystyle\int_\gamma \frac{z\,dz}{(z^2 - z(1+i) + i)}$ where γ is the circle of centre 0 and radius 2. (10)

[50]

SECTION B

2. Let $f(z) = \dfrac{1}{(e^z - 1)}$.

   (a) Show that $f(z)$ has a simple pole at $z = 0$. (7)

   (b) Let
   $$f(z) = \frac{b}{z} + a_0 + a_1 z + a_2 z^2 + \ldots$$
   be the Laurent expansion of f around 0. Compute the terms b, a_0 and a_1. (12)

   (c) Evaluate $\int_\gamma f(z)dz$ where γ is a circle of centre 0 and radius $r > 0$, traversed once anticlockwise. (6)

   [25]

3. Let $f(z) = \dfrac{z^2}{1+z^4}$.

   (a) Determine the poles of F which lie in the upper-half plane $H = \{z : \mathrm{Im}(z) > 0\}$ and compute the residues of f at these poles. (9)

   (b) Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, dx = \frac{\pi}{\sqrt{2}}$. (16)

   [25]

4. (a) Suppose that f is an analytic function on the open disc $D = \{z : |z| < 1\}$ and that $\mathrm{Re}(f(z)) = 3$ for all $z \in D$. Show that f is constant in D. (5)

   (b) Let f be analytic inside and on a simple closed curve γ. Suppose that $f = 0$ on γ. Show that $f = 0$ inside γ. (5)

   (c) Compute the Taylor series of $F(z) = \dfrac{1}{z}$ around $z_0 = 1$ and its radius of convergence. (5)

   (d) Show that $\displaystyle\int_\gamma \frac{5z-2}{z(z-1)} dz = 10\pi i$, where γ is any circle of radius greater than 1 and centre 0. (10)

   [25]