MAS3006

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING, COMPUTER SCIENCE AND MATHEMATICS

MATHEMATICAL SCIENCES

COMPLEX ANALYSIS

June 2006

Time allowed: 2 HOURS.

Examiner: Andreas Schweizer

This is a CLOSED BOOK examination.

The mark for this module is calculated from 75% of the percentage mark for this paper plus 25% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

SECTION A

1. (a) Discuss whether the following subset of $\mathbb{C}$ is open or closed or both or neither:

$$U = \{z \in \mathbb{C} \ : \ |z| < 1, \ Im(z) \geq 0\}.$$

(8)

(b) Determine where the function

$$\frac{z^3 - 7}{e^{iz} - 1}$$

is holomorphic and calculate its derivative. (10)

(c) Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty}(n+4)z^n \quad \text{and} \quad \sum_{n=0}^{\infty}\frac{z^{2n}}{9^n}$$

(10)

(d) Evaluate the integral

$$\int_{\gamma}\frac{\sin(e^z - 1)}{z^2}dz$$

where γ is the unit circle traversed once counter-clockwise. (10)

(e) Let γ_r be the circle with center 0 and radius r, traversed once counter-clockwise. Evaluate

$$\int_{\gamma_r}\frac{e^{z+4}}{z^2 + 2z - 8}dz$$

for $r = 1$, $r = 3$ and $r = 5$. (12)

[50]

SECTION B

2. (a) In each of the following cases determine the limit or show that it does not exist:

$$\lim_{z\to 2i} \frac{5iz+10}{z^2-5iz-6};$$

and

$$\lim_{z\to 0} f(z) \text{ where } f(z) = f(x+iy) = \frac{5x^3+iy^2}{x^2+y^2}.$$

(8)

(b) Let $z = x+iy$ and $f(z) = (x^2+2xy)+i(4x+y^2)$. Show that f is not holomorphic at any point $z \in \mathbb{C}$. (7)

(c) Let f be a holomorphic function in a domain D such that $Im(f(z)) = 5Re(f(z))$ for all $z \in D$. Show that f is constant. (10)

[25]

3. (a) Determine the largest open disk around 0 on which the function

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n \cdot 3^n}$$

is analytic. Give a simple expression (not the power series) for the derivative $f'(z)$. (10)

(b) Expand the function e^{2z^2} in a power series around 0. Where does this power series converge? (5)

(c) Evaluate

$$\int_\gamma \frac{e^{2z^2}}{z^{77}} dz$$

where γ is a circle around 0, traversed once counter-clockwise. What is the value of

$$\int_\gamma z^{77} e^{2z^2} dz$$

and why? (10)

[25]

4. (a) Find the first two coefficients of the Laurent series around 0 of the function

$$\frac{1}{e^z - 1 - z}.$$

(7)

(b) Show that

$$\int_\gamma \frac{1}{e^z - 1 - z} dz = \frac{-4\pi i}{3}$$

where γ is a sufficiently small circle around 0, traversed once counter-clockwise. (3)

(c) Using residues, evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x^2 + 5}{(x^2 + 1)(x^2 + 2)} dx.$$

(15)

[25]