# MAS3006

## UNIVERSITY OF EXETER

## SCHOOL OF ENGINEERING, COMPUTER SCIENCE AND MATHEMATICS

## MATHEMATICAL SCIENCES

## COMPLEX ANALYSIS

May/June 2007

Time allowed: 2 HOURS.

Examiner: Dr. Andreas Schweizer

This is a CLOSED BOOK examination.

The mark for this module is calculated from 75% of the percentage mark for this paper plus 25% of the percentage mark for associated coursework.

**Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).**

*Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.*

## SECTION A

1. (a) Discuss whether the following subset of $\mathbb{C}$ is open or closed or both or neither:

$$U = \{z \in \mathbb{C} \,:\, |z| \leq 1,\ Re(z) > 0\}.$$

(7)

(b) Find $\lim_{n\to\infty} z_n$ for

$$z_n = \frac{2 - 3ni + (-1)^n i}{5n - i}.$$

(5)

(c) Determine where the function

$$\frac{z^2 - 2}{e^{\pi z} - 1}$$

is holomorphic and calculate its derivative. (8)

(d) Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n+3}{4^n} z^{2n}.$$

(6)

(e) What is the biggest open disk around 0 in which the function

$$f(z) = \frac{1}{5 - 3z}$$

is analytic. Find the power series around 0 of this function. (8)

(f) Let $\gamma$ be the following contour: straight line from 1 to 2, followed by three-quarter-circle with centre 0 from 2 to $-2i$, followed by straight line from $-2i$ to 1. Find the values of the integrals

$$\int_\gamma e^{2z} dz \quad \text{and} \quad \int_\gamma \frac{e^{2z}}{z^2} dz.$$

(6)

(g) Let $a, b \in \mathbb{C}$ with $|a| < |b|$. Let $\gamma_R$ be the circle with centre 0 and radius $R$, traversed once counter-clockwise. Evaluate

$$\int_{\gamma_R} \frac{1}{(z-a)(z-b)} dz$$

for $R < |a|$, for $|a| < R < |b|$, and for $R > |b|$. (10)

**[50]**

## SECTION B

2. (a) Does

$$\lim_{z \to 0} \frac{Re(z)}{z}$$

exist? Justify your answer. (4)

(b) Let $f(z) = f(x + iy) = xy$. Show that $f$ is not differentiable at any point $z_0 \neq 0$, and that $f$ is differentiable at $z_0 = 0$. (9)

(c) Using residues, evaluate the real integral

$$\int_{-\infty}^{\infty} \frac{x^2 + 1}{(x^2 + 9)^2} dx.$$

(12)

**[25]**

3. (a) Find all holomorphic functions $f(z) = f(x + iy)$ with

$$Re(f) = 5x^3 - 15xy^2 + e^{2x} \cos(2y).$$

(7)

(b) Let $f$ be an entire function with

$$|f(z)| \leq |z| \text{ for all } z \in \mathbb{C}.$$

Show that $f(z) = \alpha z$ where $\alpha$ is a complex constant with $|\alpha| \leq 1$. (9)

(c) Using residues, evaluate the integral

$$\int_0^{2\pi} \frac{1}{5 + 3\sin\vartheta} d\vartheta.$$

(9)

**[25]**

4. (a) Classify the singularities (removable, pole or essential) at 0 of the following functions. In case of a pole give the order.

(i)
$$\frac{z^2}{\cos(z)-1}$$

(ii)
$$\cos(\frac{1}{z^2})$$

(iii)
$$\frac{\sin(z)}{z^2}$$

(9)

(b) Find the first 3 terms of the Laurent series around 0 of the function
$$f(z) = \frac{e^{2z}-1}{e^{z^2}-1}$$
What is the residue of $f$ at 0? (8)

(c) Using Rouché's Theorem, determine the number of zeroes (counted with multiplicities) of the polynomial
$$p(z) = z^5 + 5z^4 + 20z^3 + 3$$
in the disk $D(0,2)$. (8)

[25]