MAS3006

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING, COMPUTING AND MATHEMATICS

MATHEMATICAL SCIENCES

May/June 2008

COMPLEX ANALYSIS

Module Leader: Dr. Andreas Schweizer

Duration: 2 HOURS.

The mark for this module is calculated from 75% of the percentage mark for this paper plus 25% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

1. (a) Discuss whether the following subset of $\mathbb{C}$ is open or closed or both or neither:

$$T = \{z \in \mathbb{C} \ : \ Re(z) \geq 0,\ Im(z) \geq 0,\ Re(z) + Im(z) < 2\}.$$

(7)

(b) Find $\lim_{n\to\infty} z_n$ where

$$z_n = \frac{8n + 6ni + i^n}{3n + 4i}.$$

(5)

(c) Solve the quadratic equation

$$z^2 + (1 + 3i)z - 1 + \frac{3}{4}i = 0.$$

(9)

(d) Determine where the function

$$\frac{e^{\sin(z)}}{z^3 + z}$$

is holomorphic and calculate its derivative. (7)

(e) Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n+3}} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{n}{2^n} z^{2n}.$$

(8)

(f) Express the following analytic function in terms of known functions, not as a power series.

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2(n!)}$$

(No, no typo! And no, it's not the cosine!) (4)

(g) Let γ_c be the circle with centre c and radius 1, traversed once counterclockwise. Calculate

$$\int_{\gamma_c} \frac{e^z}{z(z-2)^2} dz$$

for $c = -2$, $c = 0$, and $c = 2$. (10)

[**50**]

SECTION B

2. (a) At which points is the function

$$f(z) = f(x+iy) = x^2 + ixy = zRe(z)$$

differentiable? (6)

(b) Find the first 3 terms of the Laurent series around 0 of the function

$$f(z) = \frac{e^{3z}-1}{e^{z^2}-1}$$

What is the residue of f at 0? (7)

(c) Using residues, evaluate the real integral

$$\int_{-\infty}^{\infty} \frac{x^2+3}{(x^2+4)^2} dx.$$

(12)

[25]

3. (a) Let f be a holomorphic function in a domain D such that $Re(f) - Im(f)$ is constant in D. Show that f is constant. (6)

(b) Let $M \in \mathbb{R}$ and let f be an entire function with

$$|f(z)| \leq |z|^{\frac{3}{2}} \quad \text{for all} \quad z \geq M.$$

Show that $f(z)$ is a polynomial of degree at most 1. (9)

(c) Using residues, evaluate the integral

$$\int_0^{2\pi} \frac{1}{5 - 3\sin\vartheta} d\vartheta.$$

(10)

[25]

4. (a) A rope hanging between two pillars forms the curve

$$\gamma(t) = t + i(3 + \frac{e^t + e^{-t}}{2}), \quad -1 \le t \le 1.$$

Calculate the length of the rope. (8)

(b) Classify the singularities (removable, pole or essential) at 0 of the following functions. In case of a pole give the order.

(i)
$$\frac{\cos(z) - 1}{\sin(z) - z}$$

(ii)
$$\frac{\cos(z) - z}{\sin(z) - 1}$$

(iii)
$$\cos(\frac{1}{z}) + \sin(\frac{1}{z})$$

(9)

(c) Using Rouché's Theorem, determine the number of zeroes (counted with multiplicities) of the polynomial

$$p(z) = z^4 + 4z^3 + z^2 + 2z + 5$$

in the disk $D(0, 3)$. (8)

[25]