ECM3703

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING, COMPUTING AND MATHEMATICS

MATHEMATICAL SCIENCES

May/June 2009

Complex Analysis

Module Leader: Robin Chapman

Duration: 2 HOURS.

The mark for this module is calculated from 80% of the percentage mark for this paper plus 20% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

1. (a) Solve the quadratic equation

$$z^2 - (1+2i)z - 7 + i = 0.$$

(4)

(b) Determine whether the set A is open, closed or neither, justifying your answer:

$$A = \{z \in \mathbf{C} : \mathrm{Re}(z) > \mathrm{Im}(z) \geq 0\}.$$

(8)

(c) Using the Cauchy-Riemann equations, or otherwise, find a holomorphic function f on $\mathbf{C}$ satisfying

$$\mathrm{Re}(f(x+yi)) = x \sin x \cosh y - y \cos x \sinh y.$$

(10)

(d) Determine where the function

$$g(z) = \frac{\cos(1/z)}{z^2+2}$$

is holomorphic and find its derivative there. (5)

(e) Find the radius of convergence of the power series

$$h(z) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!(2n)!} z^n.$$

(6)

(f) Let γ denote the circular contour with centre 0 and radius 2, traversed anticlockwise. Prove that

$$\left| \int_\gamma \frac{e^z \, dz}{z^2(z^2+1)} \right| \leq \frac{\pi e^2}{3}.$$

(6)

(g) Let Γ_a denote the circular contour with centre a and radius 1, traversed anticlockwise. Calculate the contour integral

$$\int_{\Gamma_a} \frac{\sin(\pi z)}{z^2(z-2)} \, dz$$

for $a = -2$, for $a = 0$ and for $a = 2$. (11)

[**50**]

SECTION B

2. (a) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+1)(x^2+2)}.$$

(10)

(b) By integrating the function

$$f(z) = \frac{1}{z^4 \tan(\pi z)}$$

over the contour S_N, the square with vertices $(N+\frac{1}{2})(\pm 1 \pm i)$ where N is a positive integer, evaluate the infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

(You may assume without proof that

$$|\tan(x+iy)|^2 = \frac{\sin^2 x + \sinh^2 y}{\cos^2 x + \sinh^2 y}$$

for real x and y). (15)

[**25**]

3. (a) Assuming Cauchy's integral formula for the n-th derivative, prove that if f is holomorphic on $\mathbf{C}$, $a \in \mathbf{C}$, n is a nonnegative integer and $R > 0$ then

$$|f^{(n)}(a)| \leq \frac{n!}{R^n} \max\{|f(z)| : |z-a| = R\}.$$

(9)

(b) Assuming part (a) above, prove that if f is holomorphic on $\mathbf{C}$ and satisfies

$$|f(z)| \leq |z|^2 + |z|$$

for all $z \in \mathbf{C}$, then there are α, $\beta \in \mathbf{C}$ such that

$$f(z) = \alpha z^2 + \beta z$$

for all $z \in \mathbf{C}$. Also prove that $|\alpha| \leq 1$ and $|\beta| \leq 1$. (16)

[**25**]

4. (a) Find the terms up to the z^3 term of the Laurent series around 0 of

$$f(z) = \frac{1 - \cos z}{z - \sin z}.$$

(9)

(b) Classify the singularities (as removable, essential or poles) of the function

$$g(z) = \frac{\sin\left(\dfrac{z}{z+1}\right)}{z(z-1)^2(z^2+1)}$$

giving the order of each pole. (9)

(c) Using Rouché's theorem, calculate the number of zeros (counted with multiplicity) of

$$h(z) = z^5 - 5z^4 + z^2 + 9z - 1$$

in the annulus with $1 < |z| < 3$. (7)

[**25**]