ECM3703

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING, COMPUTING AND MATHEMATICS

MATHEMATICAL SCIENCES

May/June 2010

COMPLEX ANALYSIS

Module Leader: Dr P. J. Truman

Duration: 2 HOURS.

The mark for this module is calculated from 80% of the percentage mark for this paper plus 20% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

1) (a) Sketch the following subset of $\mathbb{C}$ and discuss whether it is open, closed or neither:

$$\{z \in \mathbb{C} \mid 1 \leq |z| < 2 \text{ and } \operatorname{Re}(z) + \operatorname{Im}(z) \geq 0\}.$$

[7]

(b) Find the radius of convergence of the following series:

$$\sum_{n=0}^{\infty} \frac{i^n (n!)^2}{(2n)!} z^n.$$

[7]

(c) Let γ be the circle with centre 0 and radius 1, traversed once anitclockwise. Using the estimation theorem, prove that

$$\left| \int_\gamma \frac{\cos z}{2z^2}\, dz \right| \leq \pi e.$$

[8]

(d) Using the Cauchy Riemann Equations, or otherwise, find a holomorphic function f on $\mathbb{C}$ satisfying

$$\operatorname{Re}(f(x+iy)) = \sinh(x^2 - y^2)\cos(2xy).$$

Give your solution in the form $f(z)$. [10]

(e) Let γ be the circle with centre 1 and radius 1, traversed once anticlockwise. Evaluate

$$\int_\gamma \frac{iz}{z^2 - (1+i)z + i}\, dz,$$

simplifying your answer as much as possible. [8]

(f) Let γ_r be the circle with centre 0 and radius r, traversed once anticlockwise. Evaluate

$$\int_{\gamma_r} \frac{1}{z^2 - 2z + 2}\, dz$$

when

(i) $r = 1$

(ii) $r = 2$.

[10]

[50]

SECTION B

2) Let f be a complex function defined on an open set $G \subseteq \mathbb{C}$ and let $z = x + iy \in G$. Write $f(z) = u(x, y) + iv(x, y)$, where u and v are real valued functions defined on G.

(a) State the Cauchy-Riemann equations and show that if f is differentiable at z then the partial derivatives of u and v satisfy them. [10]

(b) Prove that the function $\mathrm{Im}(z^2)$ is not differentiable at any point $z \in \mathbb{C}$. [5]

(c) Let $f : \mathbb{C} \to \mathbb{C}$ be the function defined by $f(0) = 0$ and $f(z) = z^5/|z|^4$ elsewhere. By expressing $f(x + iy)$ in the form $u(x, y) + iv(x, y)$, show that f satisfies the Cauchy Riemann equations at $z = 0$ but is not differentiable there. [10]

[25]

3) (a) State Laurent's Theorem. [3]

(b) Compute the Laurent expansions of the function

$$f(z) = \frac{1}{(z+1)^2(z-3)}$$

in each of the following annular domains:

(i) $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$.

(ii) $\{z \in \mathbb{C} \mid 1 < |z| < 3\}$.

(iii) $\{z \in \mathbb{C} \mid |z| > 3\}$.

(You may use standard series expansions provided you state clearly why they are valid) [12]

(c) Let $f(z) = e^{z+1/z}$. You may assume that $f(z)$ is holomorphic except at $z = 0$. Let

$$\sum_{n=-\infty}^{\infty} c_n z^n$$

be the Laurent expansion of $f(z)$ around 0. Write down an expression for the coefficient c_n and deduce that

$$\int_0^{2\pi} e^{2\cos t}\, dt = 2\pi \sum_{n=0}^{\infty} \frac{1}{(n!)^2}.$$

[10]

[25]

4) (a) State Cauchy's Residue Theorem. [5]

(b) Determine the poles of

$$f(z) = \frac{1}{(z^2+1)(z^2+4)}$$

and calculate the residues at these poles. [8]

(c) Show that

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\, dx = \frac{\pi}{6}.$$

[12]

[**25**]