ECM3703

UNIVERSITY OF EXETER

COLLEGE OF ENGINEERING, MATHEMATICS AND PHYSICAL SCIENCES

MATHEMATICS

May/June 2011

Complex Analysis

Module Leader: Robin Chapman

Duration: 2 HOURS.

The mark for this module is calculated from 80% of the percentage mark for this paper plus 20% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

1. (a) Solve the quadratic equation

$$z^2 - (8+i)z + 18 + 4i = 0.$$

(3)

(b) Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n!(3n)!}{((2n)!)^2} z^n.$$

(5)

(c) Determine whether the set

$$A = \{z \in \mathbf{C} : \text{Re}(z) \geq 0, \text{Im}(z) \geq 0 \text{ and } |z| < 1\}$$

is open, closed or neither. (6)

(d) By using the Cauchy-Riemann equations, or otherwise, find a function f holomorphic on $\mathbf{C}$ with

$$\text{Re}(f(x+iy)) = x \cos x \cosh y + y \sin x \sinh y.$$

(10)

(e) Prove that

$$\left| \int_\gamma \frac{\sin(z^2)}{z^3}\, dz \right| \leq 2\pi e$$

where γ is the circle with centre 0 and radius 1. (6)

(f) Find the Laurent series expansion of the function

$$f(z) = \frac{1}{(z-1)(z+2)}$$

valid for $1 < |z| < 2$. (8)

(g) Let γ_a denote the circle with centre a and radius 1 traversed anticlockwise. Compute

$$\int_{\gamma_a} \frac{e^{-z}}{z^2(z+2)}\, dz$$

for (i) $a = -2$, (ii) $a = 0$ and (iii) $a = 2$. (12)

[**50**]

SECTION B

2. Let
$$f(z) = \frac{e^z + 1}{e^z - 1}.$$

   (a) Prove that every singularity of f is a simple pole, and find the residue of f at each pole. (6)

   (b) Consider the Laurent expansion of f at 0:
$$f(z) = \frac{a}{z} + \sum_{n=0}^{\infty} b_n z^n.$$
   Prove that $b_n = 0$ whenever n is even, and compute a, b_1 and b_3. (14)

   (c) Evaluate $\int_\gamma f(z)\, dz$ where γ is the circle with centre 0 and radius 10, traversed anticlockwise. (5)

[25]

3. (a) Using the calculus of residues, evaluate the integral
$$\int_{-\infty}^{\infty} \frac{dx}{\cosh x}.$$
   (You may assume that $|\cosh(x + iy)|^2 = \sinh^2 x + \cos^2 y$ for real x and y.) (10)

   (b) Evaluate the integral
$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^4 + 4}\, dx.$$
(15)

[25]

4. (a) Let f be holomorphic on $\mathbf{C}$. Assuming Cauchy's integral formula for the n-th derivative of f, prove Cauchy's estimate:

$$|f^{(n)}(a)| \leq \frac{n!}{r^n} \max\{|f(z)| : |z-a| = r\}.$$

(5)

(b) Let f be holomorphic on $\mathbf{C}$. Prove that if

$$|f(z)| \leq |z|^2 + |z|^3$$

for all $z \in \mathbf{C}$ then, $f(z) = \alpha z^2 + \beta z^3$ where α and β are complex constants. Moreover prove that $|\alpha| \leq 1$ and $|\beta| \leq 1$. (13)

(c) How many zeros, counted with multiplicity, does the polynomial

$$z^5 + z^4 - 10z^3 + 15z + 1$$

have in the annulus $\{z \in \mathbf{C} : 1 < |z| < 2\}$. (7)

[**25**]