ECM3703

UNIVERSITY OF EXETER

COLLEGE OF ENGINEERING, MATHEMATICS AND PHYSICAL SCIENCES

MATHEMATICS

May/June 2012

Complex Analysis

Module Leader: Robin Chapman

Duration: 2 HOURS.

The mark for this module is calculated from 80% of the percentage mark for this paper plus 20% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

1. (a) Find all complex solutions of the equation

$$z^3 + (6 + 5i)z + 5 - 5i = 0$$

given that $z = 1 - 3i$ is one solution. (10)

(b) Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(2-i)^n (2n)!}{(n!)^2} z^n.$$

(6)

(c) By using the Cauchy-Riemann equations, or otherwise, find a function f holomorphic on $\mathbf{C}$ with

$$\mathrm{Re}(f(x + iy)) = x^4 + x^3 y - 6x^2 y^2 - xy^3 + y^4.$$

(8)

(d) Find the Laurent series expansion of the function

$$f(z) = \frac{1}{(z+1)(z+3)}$$

which is valid in the annulus where $1 < |z| < 3$. (8)

(e) Let γ_a denote the circle with centre a and radius $1/2$. Compute

$$\int_{\gamma_a} \frac{\exp(z^2)}{(z-1)(z^2-1)}\, dz$$

for (i) $a = -1$, (ii) $a = 0$ and (iii) $a = 1$. (10)

(f) How many zeros, counted with multiplicity, has the function

$$g(z) = z^4 - 6z^3 - 3z^2 + 12z - 1$$

in the annulus $\{z \in \mathbf{C} : 1 < |z| < 3\}$? (8)

[50]

SECTION B

2. Let

$$f(z) = \frac{1}{z(e^z - 1)}.$$

(a) Prove that every singularity of f is a pole, and find the order of each pole. (6)

(b) Consider the Laurent expansion of f at 0:

$$f(z) = \sum_{n=-N}^{\infty} b_n z^n$$

where N is the order of the pole of f at 0. Find all the b_n for $-N \leq n \leq 1$. (10)

(c) Evaluate $\int_\gamma f(z)\, dz$ where (i) γ is the unit circle (with centre 0 and radius 1), and (ii) γ is the circle with centre $3i$ and radius 4. (9)

[**25**]

3. (a) Evaluate the integral

$$\int_0^\infty \frac{x^5}{x^{12} + 1}\, dx.$$

(12)

(b) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2 + 1)^2}\, dx.$$

(13)

[**25**]

4. (a) Let f be an entire function (a function holomorphic on all of $\mathbf{C}$.) Assuming Cauchy's integral formula for the n-th derivative of f, prove Cauchy's estimate:

$$|f^{(n)}(a)| \leq \frac{n!M}{r^n}$$

where $M = \max\{|f(z)| : |z-a| = r\}$.
Deduce that if f is entire and

$$|f(z)| \leq |z| + |z|^3$$

for all $z \in \mathbf{C}$, then there are α, β and $\gamma \in \mathbf{C}$ with

$$f(z) = \alpha z + \beta z^2 + \gamma z^3.$$

Moreover, prove that $|\alpha| \leq 1$, $|\beta| \leq 2$ and $|\gamma| \leq 1$. (15)

(b) Let $g(z) = \exp((z + z^{-1})/2)$. By computing c_0 in the Laurent series

$$g(z) = \sum_{m=-\infty}^{\infty} c_m z^m,$$

and performing a suitable contour integral, prove that

$$\int_0^{2\pi} e^{\cos t}\, dt = 2\pi \sum_{n=0}^{\infty} \frac{1}{4^n (n!)^2}.$$

(10)

[**25**]