ECM3703

UNIVERSITY OF EXETER

COLLEGE OF ENGINEERING, MATHEMATICS AND PHYSICAL SCIENCES

MATHEMATICS

May 2013

Complex Analysis

Module Leader: Robin Chapman

Duration: 2 HOURS.

The mark for this module is calculated from 80% of the percentage mark for this paper plus 20% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

1. (a) Find all complex solutions of the equation

$$z^3 + 5z^2 + (9 - 5i)z + 10 - 10i = 0$$

given that $z = 2i$ is one solution. (6)

(b) Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(2+i)^n (n!)^2}{(1+i)^n (2n)!} z^n.$$

(4)

(c) Let f be holomorphic and write

$$f(x + iy) = u(x, y) + iv(x, y).$$

Assuming the Cauchy-Riemann equations, prove that

$$f' = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.$$

Hence, or otherwise, find a function f holomorphic on $\{z \in \mathbf{C} : z \neq 0\}$ with

$$u(x, y) = \frac{x + y}{x^2 + y^2}$$

and $f(1) = 1$. (12)

(d) Find the Laurent series expansion of the function

$$f(z) = \frac{z}{(z + 1)(z - 2)}$$

which is valid on the annulus $\{z \in \mathbf{C} : 1 < |z| < 2\}$. (10)

(e) Let γ_a denote the triangular contour with vertices $-1 + i$, $-1 - i$ and a traversed in the anticlockwise direction. Compute

$$\int_{\gamma_a} \frac{\cos(\pi z)}{(z - 1)(z - 3)^2}\, dz$$

for (i) $a = 0$, (ii) $a = 2$ and (iii) $a = 4$. (12)

(f) How many zeros, counted with multiplicity, has the function

$$f(z) = e^z - 3z^{2013}$$

on the disc $\{z \in \mathbf{C} : |z| < 1\}$? (6)

[50]

SECTION B

2. Let

$$f(z) = \frac{1}{z(e^z + 1)}.$$

(a) Prove that every singularity of f is a pole. Find the order of each pole of f and also the residue of f there. (8)

(b) Consider the Laurent expansion of f at 0:

$$f(z) = \sum_{n=-N}^{\infty} b_n z^n$$

where N is the order of the pole of f at 0. Find all the b_n for $-N \leq n \leq 2$. (7)

(c) Evaluate $\int_\gamma f(z)\, dz$ where (i) γ is the unit circle (with centre 0 and radius 1), and (ii) γ is the circle with centre $5i$ and radius 6. (10)

[**25**]

3. (a) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x^2 \cos x}{(x^2+1)^2}\, dx.$$

(10)

(b) By integrating

$$f(z) = \frac{1}{z^3 \cos(\pi z)}$$

over the boundary S_N of the square with vertices at $N(1+i)$, $N(-1+i)$, $N(-1-i)$ and $N(1-i)$, and letting $N \to \infty$, prove that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

(You may assume, without proof, the identity

$$|\cos(x+iy)|^2 = \cos^2 x + \sinh^2 y$$

valid for real x and y.) (15)

[**25**]

4. (a) Let f be an entire function (a function holomorphic on all of $\mathbf{C}$.) Assuming Cauchy's integral formula for the n-th derivative of f, that is

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

where C is a circle surrounding a, prove Cauchy's estimate:

$$|f^{(n)}(a)| \le \frac{n!M}{r^n}$$

where $M = \max\{|f(z)| : |z-a| = r\}$.
Deduce that if f is entire and

$$|f(z)| \le 1 + |z|^3$$

for all $z \in \mathbf{C}$, then there are a_0, a_1, a_2 and $a_3 \in \mathbf{C}$ with

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3.$$

Moreover, prove that $|a_1| < 19/10$ and $|a_3| \le 1$. (13)

(b) Let $U = \{z \in \mathbf{C} : z \ne 0\}$. Assuming Laurent's theorem, prove that the only holomorphic functions on U satisfying $f(2z) = f(z)$ for all $z \in U$ are the constant functions. By giving its Laurent expansion, or otherwise, prove that there is a nonzero holomorphic function g on U satisfying

$$g(2z) = \sqrt{2}zg(z).$$

(12)

[25]