Complex analysis: my exam questions

These are the questions on complex analysis that I sat in my undergraduate exams, many, many years ago.

- State Cauchy's formula for the nth derivative of a holomorphic function.

  State and prove Morera's theorem.

  Suppose f_n is a sequence of holomorphic functions on an open set Ω converging pointwise to a function f, and such that $|f(z)| \leq M$ for all $z \in \Omega$. Show that f is holomorphic on Ω and that $\frac{d^k}{dz^k} f_n$ converges to $\frac{d^k}{dz^k} f$.

- Evaluate by contour integration

  (i) $$\int_0^{2\pi} \frac{\sin^2 t}{(a + b\cos t)}\, dt, \qquad 0 < b < a.$$

  (ii) $$\int_0^{\infty} \frac{(\log t)^n}{1 + t^4}\, dt, \qquad n = 0, 1, 2.$$

- Let $D = \{z \in \mathbf{C} : |z| < 1\}$ and suppose that f is holomorphic on an open neighbourhood of $\overline{D}$. Show that each of the following implies that f is constant on $\overline{D}$.

  (i) $f'(z) = 0$ whenever $z \in D$,

  (ii) $|f(z)|$ is constant for $z \in D$,

  (iii) $f(z_n)$ is constant where $\{z_n : n \in \mathbf{N}\} \subset D$ is a sequence converging to 0,

  (iv) $|f(0)| = \sup\{|f(z)| : z \in \overline{D}\}$,

  (i) $|f(0)| \geq \sup\{|f(z)| : |z| = 1\}$.

- (a) Show that the equation

  $$\left| \frac{z - \alpha}{z - \beta} \right| = \lambda$$

  where $\lambda \in \mathbf{R}$, $\lambda > 0$ and α, $\beta \in \mathbf{C}$ with $\alpha \neq \beta$ represents a circle or straight line, and that, conversely, any circle or straight line may be so represented.

  Define a Möbius transformation, and show that the image in $\mathbf{C}$ of a cicrle or straight line in $\mathbf{C}$ under a Möbius is either a circle, or a circle woth one point removed, or a straight line.

(b) Find a conformal mapping of $\{z : |z| < 1\} \cap \{z : |z - \frac{1}{2}| > \frac{1}{2}\}$ onto an annulus.

RJC 5/3/2013