

Power series

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1 Convergence

A power series with centre $a \in \mathbf{C}$ is a series of the form

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n.$$

We first consider for which values of z the series converges. Certainly $f(z)$ converges for $z = a$ since then $(z-a)^n = 0$ for all $n \geq 1$. It is possible that it converges for no values of z save for $z = a$, that it converges for all values of z , or that it converges for some values of z apart from a but not all of them. In general the series $f(z)$ has a *radius of convergence* R such that when $|z-a| < R$ it converges and when $|z-a| > R$ it diverges. When $f(z)$ converges for all z we say that $R = \infty$.

Theorem 1 *The power series*

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$$

has a radius of convergence R which is either a real number with $R \geq 0$ or is ∞ such that $f(z)$ converges absolutely when $|z-a| < R$ and diverges if $|z-a| > R$.

Proof We remark that by writing $z-a = w$ we get the power series

$$f(w+a) = \sum_{n=0}^{\infty} c_n w^n$$

which has centre 0. It makes no loss of generality to assume that $a = 0$, that is

$$f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

We first claim that if $f(z_0)$ converges for some nonzero z_0 then $f(z)$ converges absolutely for all z with $|z| < |z_0|$. Since

$$f(z_0) = \sum_{n=0}^{\infty} c_n z_0^n$$

is convergent, its terms must tend to zero. Consequently $c_n z_0^n \rightarrow 0$ as $n \rightarrow \infty$. It follows that the sequence $(c_n z_0^n)$ is bounded, that is there is some $C > 0$ such that $|c_n z_0^n| \leq C$ for all n .

Let $z \in \mathbf{C}$ satisfy $|z| < |z_0|$. Then

$$|c_n z^n| = |c_n z_0^n| \left| \frac{z}{z_0} \right|^n \leq C \left| \frac{z}{z_0} \right|^n.$$

As $|z/z_0| < 1$ the series

$$\sum_{n=0}^{\infty} |c_n z^n|$$

converges by comparison to the geometric series

$$C \sum_{n=0}^{\infty} \left| \frac{z}{z_0} \right|^n.$$

Hence $f(z)$ is absolutely convergent for all $z \in D(0, |z_0|)$.

As a consequence of this, we claim that if $f(z_1)$ diverges and $|z| > |z_1|$ then $f(z)$ also diverges. For if $f(z)$ converges then $f(z_1)$ must converge, since $|z_1| < |z|$, a contradiction.

Define R to be the least upper bound of the set

$$\mathcal{A} = \{|z_0| : f(z_0) \text{ converges}\}.$$

As $0 \in \mathcal{A}$ the set $\mathcal{A}$ is nonempty. Also when $\mathcal{A}$ is unbounded we set $R = \infty$. Certainly if $|z| > R$ then $z \notin \mathcal{A}$ and the series $f(z)$ diverges.

Suppose that $|z| < R$. Then $|z|$ is not an upper bound for $\mathcal{A}$ so there is $z_0 \in \mathbf{C}$ such that $f(z_0)$ converges and $|z| < |z_0|$. It follows that $f(z)$ converges absolutely. $\square$

The moral of this tale is that the radius of convergence behaves the same way as it does for power series over the reals. Indeed the complex series $\sum_{n=0}^{\infty} a_n z^n$ has the same radius of convergence as the real series $\sum_{n=0}^{\infty} |a_n| x^n$. This means we can use familiar techniques for finding the radius of convergence. Most examples in practice can be resolved using the ratio test

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

and a few more yield to the root test

$$\frac{1}{R} = \lim_{n \rightarrow \infty} |a_n|^{1/n}$$

(provided these limits exist). There is a formula that **always** works:

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{1/n}.$$

However this uses the notion of the “limsup” which I don’t think comes up in the Exeter syllabus (it’s a topic I could have treated in my second year Analysis lectures, but didn’t...).

2 Differentiation

We prove the following “obvious” theorem involving power series.

Theorem 2 *Let*

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$$

be a power series convergent on the open disc $D(a, R)$. Then f is holomorphic on $D(a, R)$; moreover

$$f'(z) = \sum_{n=1}^{\infty} c_n(z-a)^{n-1}.$$

Proof Again, as in the proof of theorem 1, we may assume that $a = 0$, that is

$$f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

From theorem 1 it follows that if $f(z)$ convergent on an open disc $D(0, r)$ then it is absolutely convergent there. In particular

$$\sum_{n=0}^{\infty} |c_n| s^n$$

is convergent whenever $0 \leq s < R$.

Let $z_0 \in D(0, R)$. We aim to prove that f is differentiable at z_0 . To this end we show that f is given by a power series convergent in $D(z_0, r)$ for a suitable r . As $|z_0| < R$ there is a number r with $r > 0$ and $|z_0| + r = s < R$. We aim to find a power series

$$\sum_{k=0}^{\infty} d_k(z-z_0)^k$$

which converges to $f(z)$ on $D(z_0, r)$.

Note that if $z \in D(z_0, r)$, then $z \in D(0, R)$ and so

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_n ((z - z_0) + z_0)^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n c_n \binom{n}{k} (z - z_0)^k z_0^{n-k}. \end{aligned}$$

We now reverse the order of summation in this double sum. But beware: **it is not always true that reversing the order of summation leaves the value of a double sum unchanged**. However in this case we are safe since the double sum is absolutely convergent (see the discussion in the appendix). The sum is absolutely convergent as

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^n \left| c_n \binom{n}{k} (z - z_0)^k z_0^{n-k} \right| &= \sum_{n=0}^{\infty} |c_n| \sum_{k=0}^n \binom{n}{k} |z - z_0|^k |z_0|^{n-k} \\ &= \sum_{n=0}^{\infty} |c_n| (|z - z_0| + |z_0|)^n \\ &\leq \sum_{n=0}^{\infty} |c_n| (r + |z_0|)^n \\ &= \sum_{n=0}^{\infty} |c_n| s^n \end{aligned}$$

which is convergent. Reversing the order of summation in the original series gives

$$f(z) = \sum_{n=0}^{\infty} \sum_{k=0}^n c_n \binom{n}{k} (z - z_0)^k z_0^{n-k} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} c_n \binom{n}{k} (z - z_0)^k z_0^{n-k}$$

so that for $z \in D(z_0, r)$,

$$f(z) = \sum_{k=0}^{\infty} d_k (z - z_0)^k$$

where

$$d_k = \sum_{n=k}^{\infty} c_n \binom{n}{k} z_0^{n-k}.$$

In particular

$$d_0 = \sum_{n=0}^{\infty} c_n \binom{n}{0} z_0^{n-0} = \sum_{n=0}^{\infty} c_n z_0^n = f(z_0)$$

and

$$d_1 = \sum_{n=1}^{\infty} c_n \binom{n}{1} z_0^{n-1} = \sum_{n=1}^{\infty} n c_n z_0^{n-1}$$

which is what we expect $f'(z_0)$ to be.

We need to prove that $f'(z_0)$ really is d_1 . For $z \in D(z_0, r)$ but $z \neq z_0$ we have

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{f(z) - d_0}{z - z_0} = \sum_{k=1}^{\infty} d_k (z - z_0)^{k-1}.$$

Consequently

$$\frac{f(z) - f(z_0)}{z - z_0} - d_1 = \sum_{k=2}^{\infty} d_k (z - z_0)^{k-1} = (z - z_0) \sum_{k=2}^{\infty} d_k (z - z_0)^{k-2}.$$

Write

$$g(z) = \sum_{k=2}^{\infty} d_k (z - z_0)^{k-2}.$$

This power series must have the same radius of convergence as

$$\sum_{k=0}^{\infty} d_k (z - z_0)^k$$

so that its radius of convergence is at least r . Thus the sum

$$\sum_{k=2}^{\infty} |d_k| (r/2)^{k-2}$$

converges to a number $C \geq 0$. If $|z - z_0| < r/2$ then

$$|g(z)| \leq \sum_{k=2}^{\infty} |d_k| |z - z_0|^{k-2} \leq \sum_{k=2}^{\infty} |d_k| (r/2)^{k-2} = C.$$

Consequently

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - d_1 \right| \leq C |z - z_0|$$

so that

$$\frac{f(z) - f(z_0)}{z - z_0} - d_1 \rightarrow 0$$

as $z \rightarrow z_0$. This means that

$$f'(z_0) = d_1 = \sum_{n=1}^{\infty} n c_n z_0^{n-1}$$

which is what we needed to prove. $\square$

Most books on complex analysis give a somewhat different proof of this theorem. The above proof gives a bonus: for each $z_0 \in D(a, R)$ the function f also equals a power series on a disc with centre z_0 .

A Appendix

Theorem 3 *Let*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{n,k}$$

be an absolutely convergent double series of complex terms. Then we may reverse the order of summation:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{n,k} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_{n,k}.$$

I shan't prove this but will discuss the result. To say the series is absolutely convergent means that for each n the single series

$$s_n = \sum_{k=0}^{\infty} |a_{n,k}|$$

is convergent and also the series

$$\sum_{n=0}^{\infty} s_n$$

is convergent.

Theorem 3 is well-known when the terms $a_{n,k}$ are real; proofs can be found in real analysis textbooks. The complex version follows readily from the real version. If

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{n,k}$$

is absolutely convergent, it follows immediately that both

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \operatorname{Re} a_{n,k} \quad \text{and} \quad \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \operatorname{Im} a_{n,k}$$

are absolutely convergent. Applying the real number version, summation in both the real parts and the imaginary parts can be reversed, which is sufficient.

Finally the reversal of summation actually used in Theorem 2 has the form

$$\sum_{n=0}^{\infty} \sum_{k=0}^n a_{n,k} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} a_{n,k}.$$

This is just the special case of Theorem 3 where $a_{n,k} = 0$ whenever $k > n$.