Some useful analytic results

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Here I give some details and results of "real analysis" type which are used and useful in the course.

Recall the Bolzano-Weierstrass theorem, which states that if (a_n) is a bounded sequence of real numbers, then (a_n) has a convergent subsequence.

Lemma (Complex Bolzano-Weierstrass)**.** *Let (z_n) be a bounded sequence of complex numbers. Then (z_n) has a convergent subsequence.*

Proof As (z_n) is bounded sequence of complex numbers, then $(\operatorname{Re} z_n)$ and $(\operatorname{Im} z_n)$ are bounded sequences of real numbers. By Bolzano-Weierstrass, $(\operatorname{Re} z_n)$ has a convergent subsequence $(\operatorname{Re} z_{n_k})$. Now the sequence $(\operatorname{Im} z_{n_k})$ is also bounded so by Bolzano-Weierstrass again, it has a convergent subsequence $(\operatorname{Im} z_{n_{k_r}})$. Now $(\operatorname{Re} z_{n_{k_r}})$ is a subsequence of the convergent sequence $(\operatorname{Re} z_{n_k})$, and so converges (to the same limit). Hence $(z_{n_{k_r}})$ is a convergent subsequence of (z_n) as its real and imaginary parts are both convergent. □

This is useful in proving the following result used in the proof of the Cauchy-Goursat theorem.

Theorem. *Let $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ be a sequence of nonempty closed bounded subsets of $\mathbf{C}$. Then there is a complex number α which is an element of all the A_m.*

Proof Pick an element $a_n \in A_n$ for each n. Then each $a_n \in A_0$, which is a bounded set. So we may apply the complex form of Bolzano-Weierstrass and deduce there is a subsequence (a_{n_k}) converging to $\alpha \in \mathbf{C}$.

Let m be a positive integer. In the sequence (a_n), all but perhaps the first m terms lie in A_m. So at most finitely many terms in the sequence (a_{n_k}) lie outside A_m. Deleting these we get a sequence of elements of A_m converging to α. As A_m is a closed set, then $\alpha \in A_m$ also (by exercise 10 of the problems sheet).

We conclude that α is an element of all of the A_m. □

Another useful consequence of Bolzano-Weierstrass is the fact that continuous functions on closed bounded sets are bounded.

Theorem. *Let A be a closed bounded subset of $\mathbf{C}$, and let $f : A \to \mathbf{C}$ be a continuous function. Then f is bounded on A, that is there is M such that $|f(z)| \le M$ for all $z \in A$.*

Proof Suppose, for sake of contradiction, that f is not bounded. Then for each positive integer n there is $z_n \in A$ with $|f(z_n)| > n$. The sequence (z_n) is bounded (as A is bounded), so it has a convergent subsequence (z_{n_k}) with limit w. By question 10 of the problem sheet, as each $z_{n_k} \in A$ then $w \in A$ (as A is closed). But then by continuity, $f(z_{n_k}) \to f(w)$ so that $|f(z_{n_k})| \to |f(w)|$ as $k \to \infty$. But $|f(z_{n_k})| > n_k$ and as $n_k \to \infty$ as $k \to \infty$ therefore $|f(z_{n_k})| \to \infty$ as $k \to \infty$, a contradiction. □

A technical result which I need in the proof of Cauchy's integral formula is that if an open set contains a closed disc then it contains a larger open disc.

Theorem. *Let U be an open subset of $\mathbf{C}$, and suppose $\overline{D}(a, r) \subseteq U$ for some $a \in \mathbf{C}$ and $r > 0$. Then there is some $s > r$ with $D(a, s) \subseteq U$.*

Proof Assume that there isn't such an s. Then for each $n \in \mathbf{N}$, s cannot be $r + 1/n$, so there is $z_n \in D(a, r + 1/n)$ with $z_n \notin U$. As $\overline{D}(a, r) \subseteq U$ then $r \le |z_n - a| < r + 1/n$. It follows that $|z_n - a| \to r$ as $n \to \infty$.

The sequence (z_n) is bounded, so it has a convergent subsequence (z_{n_k}) by Bolzano-Weierstrass. Let w denote its limit. Then $|z_{n_k} - a| \to r$ as $k \to \infty$ and so $|w - a| = r$. As $\overline{D}(a, r) \subseteq U$ then $w \in U$.

As U is open $D(w, \varepsilon) \subseteq U$ for some $\varepsilon > 0$. As $z_n \notin U$ then $|z_n - w| \ge \varepsilon$ for all n. *A fortiori* $|z_{n_k} - w| \ge \varepsilon$ for all k. This means that (z_{n_k}) cannot converge to w, a contradiction. Hence there must be an $s > r$ with $D(a, s) \subseteq U$. □