Coding Theory: Problem sheet 2

Solutions must be submitted by 12pm on Thursday 1 November 2007

1. Determine whether or not there is a $(5, 5, 3)$-code over $\{0, 1\}$ by performing the following steps:

   (i) If there is such a code, there is an equivalent code C with $00000 \in C$ so we may assume that $00000 \in C$.

   (ii) Show that $11111 \notin C$.

   (iii) Show that there is at most one word of weight 4 in C and deduce there are at least three words of weight 3 in C.

   (iv) Determine whether or not there can be three words of weight 3 in C. [10]

2. Let C be a subset of $(Z_p)^n$ for some p and n. Show that if C is closed under addition ($\mathbf{a} + \mathbf{b} \in C$ for all $\mathbf{a}$, $\mathbf{b} \in C$) then C is closed under scalar multiplication ($r\mathbf{a} \in C$ for all $r \in Z_p$ and $\mathbf{a} \in C$). [8]

3. Is the ISBN-10 code a linear code? Justify your answer. [8]

4. For each prime p with $p \leq 11$, write down the reciprocals in Z_p of each number a with $1 \leq a \leq p - 1$. [12]

5. Identify the missing symbol in the following ISBN-10s:
   (i) 074753274?, (ii) 0099?66899 (iii) 0552149?19. [9]

   (All of these are real books; you may like to identify them.)

6. Recently the thirteen-digit ISBN-13 has replaced the ten-digit ISBN-10. The thirteen digits $x_1x_2 \cdots x_{12}x_{13}$ must satisfy the congruence

   $$x_1+3x_2+x_3+3x_4+x_5+3x_6+x_7+3x_8+x_9+3x_{10}+x_{11}+3x_{12}+x_{13} \equiv 0 \pmod{10}.$$

   (i) Show that the ISBN-13 code is 1-error detecting.

   (ii) The ISBN-13 code does not detect transposition errors since swapping the first and third digits does not affect the validity of an ISBN-13. But does the ISBN-13 code detect transposition errors involving the swapping of *adjacent* digits? [10]

7. Find all codewords, and so find the minimum distances of the binary linear codes C_1 and C_2 over $Z_2 = \{0, 1\}$ with generator matrices

$$M_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

respectively. In each case find a linearly equivalent linear code with generator matrix in standard form. Are either of these codes a perfect 1-error-correcting code? [18]

8. Find all codewords, and so find the minimum distance of the ternary linear code C (with alphabet $Z_3 = \{0, 1, 2\}$) with generator matrix

$$M = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

Is C is a perfect 1-error-correcting code? [10]

9. Let C be a linear code over Z_2. Prove that either all words in C have even weight, or that exactly half of them do. [15]

[Hint: it may help to recall that the parity check code of a given length is linear over Z_2]

10. (Challenge problem) Construct fields with (i) four elements, (ii) nine elements, (iii) eight elements.

RJC 11/10/2007