Equivalence of codes

Let C be a code of length n over an alphabet Ω. There are two simple types of operation we may do to Ω^n which preserve Hamming distances. Such operations take the code C to codes with the same number of words and same minimum distances: these are called *equivalent* codes.

Type I

We permute the letters in each word in Ω^n in the same way.

For example, let $\Omega = \{0,1,2\}$ and $n = 5$, and consider the following operation: swap the first and third letters and the second and fifth letters. If we call this operation Φ, then $\Phi(01221) = 21021$ and $\Phi(10012) = 02110$. In general $d(\Phi(\mathbf{a}), \Phi(\mathbf{b})) = d(\mathbf{a}, \mathbf{b})$. If $C \subseteq \Omega^n$, then applying Φ to each word in C yields a code over Ω with the same length, the same number of words and the same minimum distance.

For example,

$$C: \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 2 & 0 & 2 \\ 1 & 2 & 0 & 1 & 0 \end{array} \longrightarrow \Phi(C): \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 2 & 0 & 2 & 2 \\ 1 & 1 & 1 & 2 & 0 \\ 2 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{array}$$

Type II

We pick a position in the code, and a permutation of the letters in Ω. Given a word $\mathbf{a} \in \Omega^n$ we apply that permutation to the letter in the given place.

For example, let $\Omega = \{0, 1, 2\}$ and $n = 5$, and apply the permutation

$$0 \mapsto 2, \quad 1 \mapsto 0, \qquad 2 \mapsto 1.$$

to the fourth letter in the word. If we call this operation Ψ, then $\Psi(01221) = 01211$ and $\Psi(10012) = 10002$. In general $d(\Psi(\mathbf{a}), \Psi(\mathbf{b})) = d(\mathbf{a}, \mathbf{b})$. If $C \subseteq \Omega^n$, then applying Ψ to each word in C yields a code over Ω with the same length, the same number of words and the same minimum distance.

For example,

$$C : \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 2 & 0 & 2 \\ 1 & 2 & 0 & 1 & 0 \end{array} \longrightarrow \Psi(C) : \begin{array}{ccccc} 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 2 & 2 & 1 & 2 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 2 \\ 1 & 2 & 0 & 0 & 0 \end{array}$$

If we can convert codes C_1 to C_2 by a sequence of operations of types I and II we say that C_1 and C_2 are *equivalent*.

Suppose that $C \subseteq \Omega^n$ is a code, and $0 \in \Omega$. Then C is equivalent to a code C' having $00 \cdots 0$ as a word. We achieve this by a sequence of operations of type II. Fix a word $\mathbf{a}$. Pick out the first nonzero entry a in $\mathbf{a}$, if it has one, and swap 0 and a in that position. Repeat, until the word becomes all-zero. For example, with $\Omega = \{0, 1, 2, 3\}$:

$$\begin{array}{ccc} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 3 \\ 3 & 0 & 2 \\ 1 & 2 & 2 \\ 3 & 1 & 1 \end{array} \rightarrow \begin{array}{ccc} 0 & 3 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 3 \\ 3 & 0 & 2 \\ 0 & 2 & 2 \\ 3 & 1 & 1 \end{array} \rightarrow \begin{array}{ccc} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 3 \\ 3 & 3 & 2 \\ 0 & 2 & 2 \\ 3 & 1 & 1 \end{array} \rightarrow \begin{array}{ccc} 0 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 3 & 2 \\ 0 & 2 & 2 \\ 3 & 1 & 0 \end{array}$$