MAS1107: Discrete Mathematics and Probability: Problem sheet 1

This problem sheet is not for assessment:
solutions will be posted on the web in due course

1. Take $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ as the universal set. Let $A = \{1, 2, 3, 5, 8\}$, $B = \{1, 5, 9\}$ and $C = \{1, 3, 7, 10\}$. Find $A \cup B$, $A \cap B$, $(A \cup B) \cap C$, $A \cup (B \cap C)$, $A \cap C^c$, $A \cap (B^c \cup C^c)$ $A \cup (B^c \cap C^c)$.

2. Write down the elements of the following sets:

$$A = \{n \in \mathbf{Z} : -5 \le n \le 5\}; \qquad B = \{n(n+1)/2 : n \in A\};$$
$$C = \{n^3 : n \in A\}; \qquad D = \{n \in A : n \text{ is odd}\}.$$

3. Let A, B and C be sets. Verify the following identities using Venn diagrams.

   (i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$; (ii) $(A \cap B)^c = A^c \cup B^c$.

4. Can you draw a suitable Venn diagram depicting four arbitrary sets?

5. 100 students were surveyed about their television viewing. Each of them admitted watching at least one of the following three programmes: *Big Brother*, *Neighbours* and *Little Britain*. 42 watch *Big Brother*, 49 watch *Neighbours*, 49 watch *Little Britain*, 13 watch *Big Brother* and *Neighbours*, 15 watch *Big Brother* and *Little Britain*, 18 watch *Neighbours* and *Little Britain*. How many watch all three programmes?

6. Let A, B and C be finite sets. Find

   (i) a formula for $|A \cap B|$ in terms of $|A|$, $|B|$ and $|A \cup B|$,

   (ii) a formula for $|A \cap B \cap C|$ in terms of $|A|$, $|B|$, $|C|$, $|A \cup B|$, $|A \cup C|$, $|B \cup C|$ and $|A \cup B \cup C|$.

7. Find the number of ways of arranging the letters A, E, M, O, U, Y in a sequence in such a way that that the words ME and YOU do not occur.

8. A bridge hand consists of thirteen cards. How many bridge hands comprise six spade cards, four diamonds, two hearts and one club?

9. A poker hand consists of five cards. They are ranked according to which combinations of values and suits they contain. One very high ranking hand is *four of a kind* containing four cards of one value (e.g.,

four Queens) and one other card. We shall work out how many such hands there are. There are 13 possibilities for the value involved in the four, and after that one can choose the other card in 48 ways, so there are $13 \times 48 = 624$ four of a kind hands. How many poker hands are there of each of the following types?

(i) *full house*: three cards of one value, and two of another, for instance, $3\spadesuit\ 3\heartsuit\ 3\clubsuit\ J\spadesuit\ J\diamondsuit$;

(ii) *three of a kind*: three cards of one value, the other two being of two other values, for instance, $5\spadesuit\ 5\heartsuit\ 5\diamondsuit\ A\spadesuit\ 7\clubsuit$;

(iii) *two pairs*: two cards of the same value, another two cards of a second value, and the last card being of a third value, for instance, $Q\heartsuit\ Q\clubsuit\ 8\spadesuit\ 8\clubsuit\ 9\diamondsuit$;

(iv) *pair*: two cards of the same value, the other three cards being of three different values, for instance, $4\spadesuit\ 4\heartsuit\ A\clubsuit\ Q\spadesuit\ J\heartsuit$.

(N.B. There are exactly $\binom{52}{5} = 2598960$ poker hands altogether, so the sum of your answers should not exceed this!)

10. Calculate Pascal's triangle up to the entries in the row with $n = 10$.

11. How many anagrams has the word SUPPRESSION? How many contain no occurrences of SPINE as a subword?

12. Sum the arithmetic progression $\sum_{n=0}^{99}(2 + 5n)$ and the geometric progression $\sum_{n=0}^{10}(-1/2)^n$.

13. Prove by induction that

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

Can you find a formula for $\sum_{k=1}^{n} k^4$? If you can, prove that too.

14. Prove by induction, or otherwise, that

$$\sum_{k=3}^{n} \binom{k}{3} = \binom{n+1}{4}.$$

Can you generalize this result?

15. Prove that if f and g are functions with the properties that $f(1) = g(1)$ and $g(n+1) - g(n) = f(n+1)$ for all $n \in \mathbf{N}$, then $\sum_{k=1}^{n} f(k) = g(n)$.

RJC 28/1/2008