MAS1107: Discrete Mathematics and Probability: Problem sheet 2

This problem sheet is not for assessment:
solutions will be posted on the web in due course

Please give numerical answers correct to three significant figures

1. A fair coin is tossed n times (that is the probability of the event A_j that it comes down heads on the j-th throw is $1/2$ and the events $A_1, \ldots, A_n$ are independent). What are the probabilities that

   (i) heads first occurs on the n-th toss?

   (ii) exactly two heads occur?

   (iii) at least two heads occur?

   (iv) exactly as many heads as tails occur?

2. Let A, B and C be events. You are given that $P(A) = 0{\cdot}4$, $P(B) = 0{\cdot}5$, $P(C) = 0 \cdot 45$, $P(A \cap B) = 0 \cdot 1.5$, $P(A \cap C) = 0 \cdot 2$, $P(B \cap C) = 0 \cdot 25$ and $P(A \cap B \cap C) = 0 \cdot 1$. Find:

   (i) $P(A \cup B)$; (ii) $P(A \cup C)$; (iii) $P(B \cup C)$; (iv) $P(A \cup B \cup C)$; (v) $P(A \cap C^c)$; (vi) $P((A \cup C) \cap B^c)$; (vii) $P(A \mid B)$; (viii) $P(A \mid C)$; (ix) $P(A \mid C^c)$; (x) $P(C \mid A)$.

3. From the inclusion-exclusion principle for two events ($P(A \cup B) = P(A) + P(B) - P(A \cap B)$) deduce the inclusion-exclusion principle for three events:

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &\quad - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &\quad + P(A \cap B \cap C). \end{aligned}$$

4. Suppose that events A and B are independent, and that $P(A) = 0 \cdot 4$ and $P(B) = 0 \cdot 6$. Find $P(A \cup B)$.

5. Prove that an event A is independent of itself if and only if $P(A) = 0$ or $P(A) = 1$.

6. Let A, B and C be events. Suppose that the three events are independent. Prove that A and $B \cup C$ are independent.

7. Three cards are drawn in sequence, without replacement, from a standard pack. Let A be the event that the first card is a spade, B be the

event that the second card is a heart, and C be the event that the third card is a king. Find:

(i) $P(A)$; (ii) $P(B)$; (ii) $P(B \mid A)$; (iv) $P(C)$; (v) $P(A \mid C)$; (vi) $P(C \mid A)$.

8. You have a bowl of twenty olives, six of which have stones, the others not. Your friend comes in and eats five of the olives at random. You then pick an olive at random and find it has a stone. What is the conditional probability that your friend picked at least one olive with a stone?

9. A product tester employed by the Gizmo Corporation of Ruritania tests the company's gizmos. A gizmo has 3% probability of being defective. The tester's test registers a defective gizmo as being defective with 95% probability, but it registers a non-defective gizmo as being defective with 2% probability. The tester performs a test on a gizmo drawn at random. Her test indicates the gizmo is defective. What is the probability that it really is defective?

10. There are three widget factories, in Aberdeen, Bradford and Cardiff. They produce respectively 25%, 35% and 40% of the Acme Widget Company's output. The failure rate of the widgets produced in the three factories are 6%, 4% and 3% respectively. A widget picked at random in the company's central warehouse is found to be defective. What is the probability that it was produced in Aberdeen? in Bradford? in Cardiff?

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