MAS1107: Discrete Mathematics and Probability: Problem sheet 3

This problem sheet is not for assessment:
solutions will be posted on the web in due course

Please give numerical answers correct to three significant figures

1. You throw five fair dice and count the number of sixes you get. Let X be the number of sixes. Tabulate $P(X = j)$ for $j = 0, 1, \ldots, 5$ and draw the graph of $F_X(a)$. (Recall that $F_X(a)$ is defined to be $P(X \leq a)$.)

2. You repeatedly draw cards at random from a standard pack until you get a spade. If you don't draw a spade you replace the card in the pack and shuffle it. Let X be the number of cards you need to draw to get a spade. Tabulate $P(X = j)$ for $j = 1, 2, \ldots, 10$ and draw the graph of $F_X(a)$ for $0 \leq a \leq 10$.

3. You keep throwing a fair die until you throw three sixes. What is the probability that the third six is thrown on the twelfth throw?

4. You deal yourself a hand of eight cards from a standard pack. Let X denote the number of spades you receive. Tabulate $P(X = j)$ for $j = 0, 1, \ldots, 8$.

5. A random variable X has Poisson distribution with parameter 2. Find $P(X \leq 2)$, $P(X = 3)$ and $P(X \geq 4)$.

6. A discrete random variable X has range space $\{1, 2, 3, 4\}$ and has $P(X = j) = j/10$ for each j in the range space. Find $E(X)$, $E(X^2)$, $E(X^3)$, $\mathrm{Var}(X)$ and $E(1/X)$.

7. A discrete random variable X has range space $\{1, 2, 3\}$ and has $P(X = j) = c/j$ for each j in the range space, where c is some suitable constant. Find c and so find $E(X)$, $E(X^2)$, $E(X^3)$, $\mathrm{Var}(X)$ and $E(1/X)$.

8. A discrete random variable X has range space $\mathbf{N} = \{1, 2, \ldots\}$. and has $P(X = j) = c/(j(j + 1))$ for each $j \in \mathbf{N}$. Conjecture and prove a formula for $P(X \leq j)$ and so find c. Prove that $E(X)$ is infinite.

9. You throw two fair dice. Let X denote the sum of the two numbers thrown. Find $E(X)$, $E(X^2)$, and $\mathrm{Var}(X)$.

10. There are n balls in a bag, numbered from 1 to n. You draw one ball and let X denote its number. Find $E(X)$, $E(X^2)$, $E(X^3)$ and $\mathrm{Var}(X)$.

11. The random variable X has mean μ and variance σ^2. What is the mean and variance of the random variable

$$Y = \frac{X - \mu}{\sigma}?$$

12. (Honour Moderations, University of Oxford, 1982) An ambidextrous student has a left and a right pocket, each initially containing n humbugs. Each time he feels hungry, he puts a hand into one of his pockets and if it is not empty, takes a humbug from it and eats it. On each occasion, he is equally likely to choose either the left or the right pocket. When he first puts his hand into an empty pocket the other pocket contains H humbugs.

    Show that if p_h is the probability that $H = h$ then

    $$p_h = \binom{2n-h}{n} \frac{1}{2^{2n-h}},$$

    and find the expected value of H, by considering

    $$\sum_{h=0}^{n} p_h, \qquad \sum_{h=0}^{n} h p_h, \qquad \sum_{h=0}^{n} (n-h) p_h$$

    or otherwise.

    [I was subjected to this exam question. You should be able to confirm the formula for p_h, but finding $E(H)$ is extremely challenging!]

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