MAS1107

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING, COMPUTER SCIENCE AND MATHEMATICS

MATHEMATICAL SCIENCES

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DISCRETE MATHEMATICS AND PROBABILITY

Module Leader: Prof T.C. Bailey

Duration: 2 HOURS.

The mark for this module is calculated from 75% of the percentage mark for this paper plus 25% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Graph Paper and Statistical Tables (Murdoch & Barnes) will be provided. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

1. (a) A card is selected at random from a fair pack of 52 playing cards. The suit of the card is noted, the card is returned to the pack and the pack is shuffled. This experiment is repeated 20 times. The random variable X is the number of times the suit of the card selected is a spade.

   (i) Specify the probability function for X and the range space of X. (3)

   (ii) Compute (using tables or otherwise) the probabilities: $P(X = 9)$; $P(X \leq 7)$ and $P(5 < X < 10)$. (6)

   (b) The random variable Y is the number of cars which pass a speed camera within a one minute period. The probability function for Y is:

$$f_Y(y) = \frac{8^y e^{-8}}{y!} \quad \text{for } y = 0, 1, 2, \ldots$$

   (i) What is the name of this probability distribution? What is the value of $\mathrm{E}[Y]$ and of $\mathrm{var}[Y]$? (4)

   (ii) Compute (using tables or otherwise) $P(3 \leq Y \leq 8)$. (2)

   (c) You are dealt (without replacement) 4 cards from a fair pack of 52 cards. Expressing your answers as decimals or as fractions in their simplest form, compute the following probabilities:

   (i) You are dealt two spades, one diamond and one club. (2)

   (ii) All four cards are of the same suit. (2)

   (iii) The first two cards dealt are spades and the last two are clubs. (3)

   (d) You roll repeatedly a fair 6-sided die. The random variable X is the number of times you have to roll it until (and including) the first time you get either a 5 or a 6.

   (i) Specify the probability function for X and the range space of X. (3)

   (ii) What is the value of $\mathrm{E}[X]$? (2)

   (iii) What is the probability that you roll the die more than three times? (2)

   (e) The **continuous** random variable Y has the probability density function:

$$f_Y(y) = \begin{cases} y - \frac{1}{2} & \text{for } 1 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

   (i) Specify in full the distribution function for Y (i.e. its cumulative probability distribution). (2)

   (ii) Compute the mean and variance of Y. (4)

   (f) The random variable X is normally distributed with mean 4 and variance 9. Compute (using tables or otherwise) the following probabilities:

   (i) $P(X < 5)$. (2)

(ii) $P(1 < X < 6)$. (3)

(iii) The value of a such that $P(|X - 4| > a) = 0.4532$. (3)

(g) A and B are two events defined on the sample space S.

(i) Specify the (mathematical) condition for A and B to be independent. (1)

(ii) Define (mathematically) the conditional probability $P(A|B)$. (1)

(iii) Specify (using the terminology of set theory) the two conditions needed for A and B to form a partition of the sample space S. (2)

(iv) If A and B form a partition of the sample space what are the values of $P(A \cup B)$ and $P(A \cap B)$? (2)

(v) Write down an expression for $P(A \cup B)$ in terms of $P(A \cap B)$, $P(A)$ and $P(B)$. (1)

[50]

SECTION B

2. (a) A sequence of independent Bernoulli trials is repeated until the r-th success is observed. The probability of success on a single trial is p. The random variable X is the total number of trials required including that on which the r-th success is observed. Determine an expression for $f_X(x)$ (the probability distribution for X), in terms of x, r and p, explaining clearly and fully how you arrive at this expression. Specify the range space of X. (8)

(b) In order to form a jury of 12 people a judge must first interview potential jurors. Assuming that 90% of all potential jurors are suitable, use the answer to part (a) to determine the probability that the judge will have to interview more than 14 potential jurors in order to find 12 who are suitable? (8)

(c) In a certain game, four boys A, B, C and D roll a fair six-sided die in turn (in the sequence A then B then C then D then A then B etc.) until one of them throws a six.

What is the probability that B wins the game on his first throw?

What is the probability that B wins the game on his second throw?

Express the probability that B wins the game as an infinite sum. Evaluate this infinite sum and hence determine the probability that B wins. (9)

[25]

3. (a) Define the covariance and the correlation between the random variables X and Y. What is meant by the statement that X and Y are uncorrelated? What is the correlation between X and X? What is the correlation between X and $-X$? (5)

(b) The random variables X and Y have the joint probability function given by the table:

		Y		
		0	1	2
	0	k	k	k
X	1	k	k	$2k$
	2	k	k	k

so, for example, $f_{XY}(1,2) = 2k$.

(i) What is the value of the constant k? (2)

(ii) What are the marginal probability distributions of X and Y? (4)

(iii) What are the mean and variance of X and of Y? (4)

(iv) What is the correlation between X and Y? (3)

(v) What is the value of $\mathrm{E}\left[2^{X+Y}\right]$? (4)

(vi) State whether or not X and Y are independent and justify your answer. (3)

[25]

4. (a) A batch of 20 items contains exactly 5 which are defective. A random sample of 8 items is taken from the batch (without replacement). The random variable X is the number of defective items in the sample. Determine an expression for $f_X(x)$ (the probability distribution for X), explaining clearly and fully how you arrive at this expression. Specify the range space of X. (6)

(b) In the National Lottery six (winning) numbers are chosen at random from the range 1 to 49 inclusive. You enter by selecting six numbers. Find the probabilities of the following events (you may leave your answers expressed in terms of permutations/combinations if you wish):

(i) Your selection matches all six of the winning numbers. (2)

(ii) Your selection matches exactly three of the winning numbers. (3)

(iii) Exactly four of the winning numbers are odd. (3)

(iv) All six winning numbers are less than 10. (3)

(v) The six winning numbers are consecutive integers. (3)

(c) A random variable X has the moment generating function

$$M_X(t) = 1/(1 - 200t) \quad \text{for all real values of } t$$

Compute the mean and variance of X. (5)

[25]